Applicable Analysis and Discrete Mathematics
available online at http://pefmath.etf.rs

Appl. Anal. Discrete Math. **11** (2017), 166–176.
DOI: https://doi.org/10.2298/AADM1701166M

THE MAXIMUM OF COTANGENT SUMS RELATED TO ESTERMANN'S ZETA FUNCTION IN RATIONAL NUMBERS IN SHORT INTERVALS

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We investigate the maximum of cotangent sums related to Estermann's zeta function in rational numbers in short intervals. This can be seen as a continuation of previous investigations of the authors on moments of these sums ([**9**], [**10**], [**11**], [**12**], [**13**]).

1. INTRODUCTION

The authors in various papers ([**9**], [**10**], [**11**], [**12**]) and the second author in his thesis [**13**], studied the distribution of cotangent sums

$$c_0\left(\frac{r}{b}\right) = -\sum_{m=1}^{b-1} \frac{m}{b} \cot\left(\frac{\pi m r}{b}\right)$$

as r ranges over the set

$$\{r \ : \ (r,b)=1,\ A_0 b \le r \le A_1 b\}\,,$$

where A_0, A_1 are fixed with $1/2 < A_0 < A_1 < 1$ and b tends to infinity.
These sums are related to the values of the Estermann zeta function $E(s, r/b, \alpha)$, which are defined by the Dirichlet series

$$E\left(s, \frac{r}{b}, \alpha\right) = \sum_{n\ge 1} \frac{\sigma_\alpha(n)\exp\left(2\pi i n r/b\right)}{n^s}\,,$$

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2010 Mathematics Subject Classification. 26A12; 11L03; 11M06.
Keywords and Phrases. Cotangent sums; Estermann's zeta function; Riemann zeta function; Riemann Hypothesis; Kloosterman sum.

where $Re\, s > Re\, \alpha + 1$, $b \geq 1$, $(r, b) = 1$ and

$$\sigma_\alpha(n) = \sum_{d|n} d^\alpha .$$

T. Estermann (see [**4**]) introduced and studied the above function in the special case when $\alpha = 0$. Much later, it was studied by I. Kiuchi (see [**8**]) for $\alpha \in (-1, 0]$. M. Ishibashi (see [**6**]) proved the following relation regarding the value of $E\left(s, \frac{r}{b}, \alpha\right)$ at $s = 0$:
Let $b \geq 2$, $1 \leq r \leq b$, $(r, b) = 1$, $\alpha \in \mathbb{N} \cup \{0\}$. Then, for even α, it holds that

$$E\left(0, \frac{r}{b}, \alpha\right) = \left(-\frac{i}{2}\right)^{\alpha+1} \sum_{m=1}^{b-1} \frac{m}{b} \cot^{(\alpha)}\left(\frac{\pi m r}{b}\right) + \frac{1}{4}\delta_{\alpha,0} ,$$

where $\delta_{\alpha,0}$ is the Kronecker delta function.
For $\alpha = 0$, the sum on the right reduces to $c_0(r/b)$.

The cotangent sum $c_0(r/b)$ can be associated to the study of the Riemann Hypothesis, also through its relation with the so-called Vasyunin sum. The Vasyunin sum is defined as follows:

$$V\left(\frac{r}{b}\right) := \sum_{m=1}^{b-1} \left\{\frac{mr}{b}\right\} \cot\left(\frac{\pi m r}{b}\right) ,$$

where $\{u\} = u - \lfloor u \rfloor$, $u \in \mathbb{R}$.
It can be shown that

$$V\left(\frac{r}{b}\right) = -c_0\left(\frac{\bar{r}}{b}\right),$$

where $\bar{r}$ is such that $\bar{r}r \equiv 1 \pmod{b}$.
The Vasyunin sum is itself associated to the study of the Riemann hypothesis through the following identity (see [**1**], [**3**]):

(1)
$$\frac{1}{2\pi(rb)^{1/2}} \int_{-\infty}^{+\infty} \left|\zeta\left(\frac{1}{2} + it\right)\right|^2 \left(\frac{r}{b}\right)^{it} \frac{dt}{\frac{1}{4} + t^2} = \frac{\log 2\pi - \gamma}{2}\left(\frac{1}{r} + \frac{1}{b}\right) + \frac{b-r}{2rb} \log\frac{r}{b} - \frac{\pi}{2rb}\left(V\left(\frac{r}{b}\right) + V\left(\frac{b}{r}\right)\right).$$

Note that the only non-explicit function in the right hand side of (1) is the Vasyunin sum. According to this approach, the Riemann Hypothesis is true if and only if

$$\lim_{N \to +\infty} d_N = 0,$$

where

$$d_N^2 = \inf_{D_N} \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left|1 - \zeta\left(\frac{1}{2} + it\right) D_N\left(\frac{1}{2} + it\right)\right|^2 \frac{dt}{\frac{1}{4} + t^2}$$

and the infimum is taken over all Dirichlet polynomials

$$D_N(s) = \sum_{n=1}^{N} \frac{a_n}{n^s}.$$

The authors in several papers ([**9**], [**10**], [**11**], [**12**]) and the second author in his thesis [**13**] investigated moments of the form

$$\frac{1}{\phi(b)} \sum_{\substack{(r,b)=1 \\ A_0 b < r \le A_1 b}} c_0\left(\frac{r}{b}\right)^{2k}, \ \frac{1}{2} < A_0 < A_1 < 1$$

and could show that

$$\frac{1}{\phi(b)} \sum_{\substack{(r,b)=1 \\ A_0 b < r \le A_1 b}} c_0\left(\frac{r}{b}\right)^{2k} = H_k b^{2k}(1+o(1)), \ (b \to +\infty),$$

where

$$H_k := \int_0^1 \left(\frac{g(x)}{\pi}\right)^{2k} dx \,,$$

$$g(x) := \sum_{l \ge 1} \frac{1 - 2\{lx\}}{l} \,.$$

The range $1/2 < A_0 < A_1 < 1$ was later extended to $0 < A_0 < A_1 < 1$ by S. Bettin in [**2**].

In this paper we investigate the maximum of

$$\left| c_0\left(\frac{r}{b}\right) \right|$$

for the values r/b in a short interval. We start with the following:

Definition 1.1. *Let* $0 < A_0 < 1$, $0 < C < 1/2$. *For* $b \in \mathbb{N}$ *we set*

$$\Delta := \Delta(b, c) = b^{-C}.$$

We set

$$M(b, C, A_0) := \max_{A_0 b \le r < (A_0 + \Delta) b} \left| c_0\left(\frac{r}{b}\right) \right| \,.$$

We shall prove the following results:

Theorem 1.2. *With Definition 1.1 let* D *satisfy* $0 < D < \frac{1}{2} - C$. *Then we have for sufficiently large* b*:*

$$M(b, C, A_0) \ge \frac{D}{\pi} b \log b \,.$$

Theorem 1.3. *Let C be as in Theorem 1.2 and let D satisfy $D > 2-C-E$, where $E \geq 0$ is a fixed constant. Let B be sufficiently large. Then we have:*

$$M(b, C, A_0) \leq \frac{D}{\pi} b \log b\,,$$

for all b with $B \leq b < 2B$ with at most B^E exceptions.

2. PRELIMINARY LEMMAS

We recall several definitions and results from [**2**].

Definition 2.4. *For $x \in \mathbb{R}$, $\mathrm{Re}(s) > 1$ we set*

$$D_{sin}(s, x) := \sum_{n \geq 1} \frac{d(n)\sin(2\pi n x)}{n^s}\,. \tag{*}$$

Lemma 2.5. *Let $\langle a_0; a_1, a_2, \ldots \rangle$ be the continued fraction expansion of $x \in \mathbb{R}$. Moreover, let u_r/v_r be the r-th partial quotient of x. Then*

$$D_{sin}(1, x) = -\frac{\pi^2}{2} \sum_{l \geq 1} \frac{(-1)^l}{v_l} \left(\left(\frac{1}{\pi v_l} \right) + \psi\left(\frac{v_{l-1}}{v_l} \right) \right), \tag{**}$$

whenever either of the two series (*), (**) is convergent.
If $x = \langle a_0; a_1, a_2, \ldots, a_r \rangle$ is a rational number then the range of summation of the series on the right is to be interpreted to be $1 \leq l \leq r$. Here ψ is an analytic function satisfying

$$\psi(x) = -\frac{\log(2\pi x) - \gamma}{\pi x} + O(\log x), \ (x \to 0)\,.$$

Lemma 2.6.

$$c_0\left(\frac{r}{b}\right) = \frac{1}{2} D_{sin}\left(0, \frac{r}{b}\right) = 2b\,\pi^{-2} D_{sin}\left(1, \frac{\bar{r}}{b}\right),$$

where $r\bar{r} \equiv 1 (\mathrm{mod}\ b)$.

Definition 2.7. *The Kloosterman sum $K(n, m, b)$ is defined by*

$$K(n, m, b) := \sum_{\substack{r=1 \\ (r,b)=1}}^{b-1} e\left(\frac{nr + m\bar{r}}{b} \right)\,.$$

For $n = 0$ (resp. $m = 0$) we obtain the Ramanujan sums $K(0, m, b)$ (resp. $K(n, 0, b)$).

Lemma 2.8. *We have the bounds*

$$|K(n,m,b)| \le \tau(b)(n,m,b)^{1/2}b^{1/2}, \tag{2.1}$$

and

$$|K(0,m,b)| \le (m,b). \tag{2.2}$$

Proof. For the result (2.1), due to Weil, (cf. [**7**]). The result (2.2) is elementary. □

Definition 2.9. *Let* Δ *be as in Definition 1.1 and* $\Omega > 0$. *We set*

$$N(b,\Delta,\Omega) := \#\{r \,:\, A_0\, b \le r < (A_0+\Delta)b,\ |\bar{r}| \le \Omega\, b\}.$$

Let the functions χ_1, χ_2 *be defined by*

$$\chi_1(u,v) := \begin{cases} 1\,, & \text{if } A_0 + v < u \le A_0 + \Delta - v \\ 0\,, & \text{otherwise}\,. \end{cases}$$

and

$$\chi_2(u) := \Delta^{-1} \int_0^{\Delta} \chi_1(u,v)dv\,.$$

Lemma 2.10. *We have*

$$\chi_2(u) = \sum_{n=-\infty}^{\infty} a(n)e(nu),$$

where $a(0) = \Delta/2$ *and*

$$a(n) = \begin{cases} O(\Delta)\,, & \text{if } |n| \le \Delta^{-1} \\ O(\Delta^{-1}n^{-2})\,, & \text{if } |n| > \Delta^{-1}\,. \end{cases}$$

Proof. The Fourier coefficients are computed as follows:

$$a(0) = \Delta^{-1} \int_0^{\Delta} \left(\int_{A_0}^{A_0+\Delta-v} 1\, du \right) dv = \frac{\Delta}{2}$$

and

$$\begin{aligned} a(n) &= \Delta^{-1} \int_0^{\Delta} \left(\int_{A_0}^{A_0+\Delta-v} e(-nu)du \right) dv \\ &= \Delta^{-1} \int_0^{\Delta} -\frac{1}{2\pi i n} \left(e(-n(A_0+\Delta-v)) - e(-nA_0)\right) dv \\ &= -\frac{\Delta^{-1}}{2\pi i n} \left(\frac{e(-nA_0)}{2\pi i n} \left(1 - e(-n\Delta)\right) - e(-nA_0) \right). \end{aligned}$$

□

Definition 2.11. *Let*

$$\chi_3(u,v) := \begin{cases} 1\,, & \text{if } \; -\Omega+v<u<\Omega \\ 0\,, & \text{otherwise}\,. \end{cases}$$

and

$$\chi_4(u) := \Omega^{-1}\int_0^{\Omega} \chi_3(u,v)dv\,.$$

Lemma 2.12. *We have*

$$\chi_4(u) = \sum_{n=-\infty}^{+\infty} c(n)\, e(nu),$$

where $c(0)=\Omega$ *and*

$$c(n) = \begin{cases} O(\Omega)\,, & \text{if } \; |n|\le \Omega^{-1} \\ O(\Omega^{-1}n^{-2})\,, & \text{if } \; |n|>\Omega^{-1}\,. \end{cases}$$

Proof. The Fourier coefficients are computed as follows

$$c(0) = \Omega^{-1}\int_0^{\Omega}\left(\int_{\Omega+v}^{\Omega-v} 1\,du\right)dv = \Omega$$

and

$$\begin{aligned} c(n) &= \Omega^{-1}\int_0^{\Omega}\left(\int_{-\Omega+v}^{\Omega-v} e(-nu)du\right)dv \\ &= \Omega^{-1}\int_0^{\Omega} -\frac{1}{2\pi i n}\left(e(-n(\Omega-v)) - e(-n(-\Omega+v)\right)dv \\ &= -\frac{\Omega^{-1}}{4\pi n^2}((2-e(n\Omega)-e(-n\Omega)))\,. \end{aligned}$$

The estimates for $c(n)$ follow by expanding $e(u)$ in its Taylor series if $|n|\le\Omega^{-1}$ and from $|e(u)|=O(1)$, if $|n|>\Omega^{-1}$. □

Lemma 2.13. *Let* $\epsilon>0$ *be such that*

$$D+\epsilon < \frac{1}{2} - C.$$

Set

$$\Omega := b^{-(D+\epsilon)}.$$

Then

$$N(b,\Delta,\Omega)>0$$

for b *sufficiently large.*

Proof. By Definitions 2.9, 2.11 we have

$$N(b,\Delta,\Omega)\geq \phi(b)a(0)c(0)+\sum_{\substack{m,n=-\infty\\(m,n)\neq(0,0)}}^{+\infty} a(m)c(n)K(n,m,b).$$

The result follows from Lemmas 2.10 and 2.12. □

Lemma 2.14. *Let $\epsilon>0$, $B\geq B(\epsilon)$, $B<b\leq 2B$. For $1\leq r<b$, $(r,b)=1$, let*

$$\frac{r}{b}=\langle 0; w_1,\ldots,w_s\rangle$$

be the continued fraction expansion of r/b with partial fractions u_i/v_i. Then there are at most 3 values of l for which

$$\frac{1}{v_l}\psi\left(\frac{v_{l-1}}{v_l}\right)\geq \log\log b$$

and at most one value of l, for which

$$\frac{1}{v_l}\psi\left(\frac{v_{l-1}}{v_l}\right)\geq \epsilon\log b.$$

Proof. Let l_i $(i=1,2,3,4)$ be such that

$$\frac{1}{v_{l_i}}\psi\left(\frac{v_{l_i-1}}{v_{l_i}}\right)\geq \log\log b.$$

Then we have

$$v_{l_1}\geq \log\log b,\ \ v_{l_2}\geq \exp(v_{l_1})\geq \log b,$$

$$v_{l_3}\geq \exp(v_{l_2})\geq b,\ \ v_{l_4}\geq \exp(v_{l_3})\geq \exp(b),$$

in contradiction to $v_s\leq b$.
In the same manner we obtain from $v_{l_j}\geq \epsilon\log b,\ j=1,2:$

$$v_s\geq \exp(\exp((\log b)^{\epsilon}))>b\,.$$

□

3. PROOF OF THEOREM 1.2

By Lemma 2.13 there is at least one

$$r\in[A_0\,b,\ (A_0+\Delta)b],$$

such that $\frac{\bar{r}}{b} \in (0, \Omega)$. By Lemmas 2.5 and 2.6 we have:

$$c_0\left(\frac{r}{b}\right) = -b \sum_{l \geq 1} \frac{(-1)^l}{v_l}\left(\left(\frac{1}{\pi v_l}\right) + \psi\left(\frac{v_{l-1}}{v_l}\right)\right).$$

Let $(u_i/v_i)_{i=1}^{s}$ be the sequence of partial fractions of $\bar{r}/b$. From

$$\Omega \geq \frac{\bar{r}}{b} \geq \frac{1}{v_1+1},$$

we obtain

$$v_1 + 1 \geq \Omega^{-1}.$$

By Lemma 2.13 we have

$$\sum_{l>1}\left(\frac{1}{\pi v_l} + \psi\left(\frac{v_{l-1}}{v_l}\right)\right) < 2\epsilon \log b, \text{ for } b \geq b_0(\epsilon).$$

Therefore,

$$\left| D_{sin}\left(0, \frac{r}{b}\right)\right| \geq \frac{1}{\pi}\log(\Omega^{-1}(1+o(1)) \quad (b \to +\infty).$$

This proves Theorem 1.2. □

4. PROOF OF THEOREM 1.3

In the sequel we assume $\epsilon > 0$ to be fixed but arbitrarily small, $Z > 0$ fixed but arbitrarily large.

Definition 4.15. *For $A_0 b < r \leq (A_0 + \Delta)b$ let*

$$\frac{\bar{r}}{b} = \langle 0; z_1, \ldots, z_w \rangle$$

be the continued fraction expansion of $\bar{r}/b$.
By Lemma 2.14 there is at most one value of l for which

$$\frac{1}{v_l}\psi\left(\frac{v_{l-1}}{v_l}\right) \geq \epsilon \log b.$$

In case of the existence of l, let

$$\frac{u_{l-1}}{v_{l-1}} := \langle 0; z_1, \ldots, z_{l-1} \rangle .$$

We write

$$u_{l-1}(r, b) = u_{l-1}$$

and

$$v_{l-1}(r,b) = v_{l-1}.$$

For s,t *with* $1 \le s,t \le Z$, $(s,t)=1$ *we write:*

$$D(s,t,l) := \left\{ (r,b) \,:\, B \le b < 2B,\ A_0 b \le r \le (A_0+\Delta)b,\ \frac{u_{l-1}(r,b)}{v_{l-1}(r,b)} = \frac{s}{t} \right\}.$$

For fixed θ *with* $0 < \theta < 1$, *let*

$$\mathcal{E}(s,t,l) := \left\{ (r,b) \,:\, B \le b < 2B,\ A_0 b \le r \le (A_0+\Delta)b,\ \frac{u_{l-1}(r,b)}{v_{l-1}(r,b)} = \frac{s}{t},\ \left|\frac{\bar{r}}{b} - \frac{s}{t}\right| \le \theta \right\}.$$

By Dirichlet's approximation theorem there is $(C_0, D_0) \in \mathbb{Z}^2$ with $1 \le D_0 \le B^2$, $(C_0, D_0) = 1$, such that

$$\left| A_0^{-1} - \frac{C_0}{D_0} \right| \le \frac{1}{D_0 B^2}\,. \tag{4.1}$$

We have

$$\sum_{B \le b < 2B} N(b,\Delta,\Omega) \le \sum_{l \le Z^2} \sum_{1 \le s,t \le Z} |\mathcal{E}(s,t,l)| \le \sum_{1 \le s,t \le Z} |\mathcal{F}(s,t)|\,, \tag{4.2}$$

where

$$\mathcal{F}(s,t) = \left\{ (b,r,\tilde{r}) \,:\, B \le b < 2B,\ A_0 b \le r \le (A_0+\Delta)b,\ \left|\frac{\tilde{r}}{b} - \frac{s}{t}\right| \le \theta,\ r\tilde{r} \equiv 1 (\mathrm{mod}\ b) \right\}.$$

We now estimate the cardinality of the set $\mathcal{F}(s,t)$.
From $A_0 b \le r \le (A_0+\Delta)b$ and (4.1) we obtain

$$b = \frac{C_0}{D_0}\, r + \frac{u}{D_0}\,, \tag{4.3}$$

where $u \in \mathbb{Z}$

$$|u| \le 4B^{1-C} D_0 \tag{4.4}$$

and

$$u \equiv -C_0 r (\mathrm{mod}\ D_0)\,. \tag{4.5}$$

The number $U(r, C_0, D_0)$ of u with (4.4), (4.5) satisfies

$$|U(r, C_0, D_0)| \ll B^{1-C}\,. \tag{4.6}$$

From (4.3) and the definition of $\tilde{r}$ we obtain

$$r\tilde{r} - 1 = y\left(\frac{C_0}{D_0}\, r + \frac{u}{D_0} \right),\ \text{with } y \in \mathbb{Z},$$

which after multiplication with C_0D_0 becomes

$$(C_0y - D_0\tilde{r})(C_0r - u) = -D_0(C_0 - \tilde{r}u)\,. \tag{4.7}$$

If $C_0 - \tilde{r}u \neq 0$ we see from the well-known estimate for the number of divisors of an integer that for a given pair $(\tilde{r}, u)$ there are at most $O(B^{\epsilon})$ pairs (r, y), such that (4.7) holds.
There are at most $O(B^{\epsilon})$ pairs $(\tilde{r}, u)$, such that $C_0 - \tilde{r}u = 0$. Thus we obtain

$$|\mathcal{F}(s,t)| = O(B^{2+\epsilon-C}\theta). \tag{4.8}$$

From (4.2) and (4.8) we obtain for $\Omega = \theta B$:

$$\sum_{B\leq b<2B} N(b, B^{1-C}, \Omega) = O(B^{2+2\epsilon-C}\theta). \tag{4.9}$$

We now apply (4.9) with $\theta = B^{-D'}$, where

$$D > D' > 2 - C - E.$$

If $\epsilon > 0$ is chosen sufficiently small we conclude from (4.9) the following:
For all b with $B \leq b < 2B$ with at most B^E exceptions we have:

$$N(b, B^{1-C}, \Omega) = 0$$

and thus

$$\frac{1}{v_l}\psi\left(\frac{v_{l-1}}{v_l}\right) \leq \frac{D'}{\pi v_{l-1}}\, b\log b\,(1 + o(1)) \tag{4.10}$$

for all $l \leq Z$.
The result of Theorem 1.3 follows now from Lemmas 2.5, 2.6 and 2.14.

Acknowledgments. The second author (M. Th. Rassias) would like to acknowledge financial support obtained through the Forschungskredit grant (Grant Nr. FK-15-106) of the University of Zurich.

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(Recived 02.01.2017.)
(Revised 03.03.2017.)

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