

On a Multidimensional Half-Discrete Hilbert-Type Inequality Related to the Hyperbolic Cotangent Function

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Abstract

In this paper, by the application of methods of weight functions and the use of analytic techniques, a multidimensional more accurate half-discrete Hilbert-type inequality with the kernel of the hyperbolic cotangent function is proved. We show that the constant factor related to the Riemann zeta function is the best possible. Equivalent forms as well as operator expressions are also investigated.

Key words: half-discrete Hilbert-type inequality; hyperbolic cotangent function; weight function; Riemann zeta function; equivalent form; Hilbert-type operator.

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1 Introduction

Assuming that $p > 1, \frac{1}{p} + \frac{1}{q} = 1, f(x), g(y) \geq 0, f \in L^p(\mathbb{R}_+), g \in L^q(\mathbb{R}_+)$,

$$\|f\|_p = \left\{ \int_0^\infty f^p(x) dx \right\}^{\frac{1}{p}} > 0,$$

$\|g\|_q > 0$, we have the following Hardy-Hilbert's integral inequality (cf. [1]):

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} dx dy < \frac{\pi}{\sin(\pi/p)} \|f\|_p \|g\|_q, \quad (1.1)$$

where the constant factor $\frac{\pi}{\sin(\pi/p)}$ is the best possible. If $a_m, b_n \geq 0, a = \{a_m\}_{m=1}^\infty \in l^p, b = \{b_n\}_{n=1}^\infty \in l^q, \|a\|_p = \{\sum_{m=1}^\infty a_m^p\}^{\frac{1}{p}} > 0, \|b\|_q > 0$, then we still have the following discrete variant of the above inequality named of Hardy-Hilbert's inequality with the same best constant $\frac{\pi}{\sin(\pi/p)}$ (cf. [1]):

$$\sum_{m=1}^\infty \sum_{n=1}^\infty \frac{a_m b_n}{m+n} < \frac{\pi}{\sin(\pi/p)} \|a\|_p \|b\|_q. \quad (1.2)$$

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For $p = q = 2$, (1.1) and (1.2) deduce to the famous Hilbert's inequalities. Inequalities (1.1) and (1.2) are important in Analysis and its applications (cf. [1], [2], [3], [4], [5], [33]).

In 1998, by introducing an independent parameter $\lambda \in (0, 1]$, Yang [6] gave an extension of (1.1) for $p = q = 2$. In 2009-2011, Yang [3], [4] gave some extensions of (1.1) and (1.2) as follows: If $\lambda_1, \lambda_2 \in \mathbb{R}$, $\lambda_1 + \lambda_2 = \lambda$, $k_\lambda(x, y) \geq 0$ is a homogeneous function of degree $-\lambda$, with $k(\lambda_1) = \int_0^\infty k_\lambda(t, 1)t^{\lambda_1-1}dt \in \mathbb{R}_+$, $\phi(x) = x^{p(1-\lambda_1)-1}$, $\psi(y) = y^{q(1-\lambda_2)-1}$, $f(x), g(y) \geq 0$,

$$f \in L_{p,\phi}(\mathbb{R}_+) = \left\{ f; \|f\|_{p,\phi} := \left\{ \int_0^\infty \phi(x)|f(x)|^p dx \right\}^{\frac{1}{p}} < \infty \right\},$$

$g \in L_{q,\psi}(\mathbb{R}_+)$, $\|f\|_{p,\phi}, \|g\|_{q,\psi} > 0$, then we have

$$\int_0^\infty \int_0^\infty k_\lambda(x, y)f(x)g(y)dxdy < k(\lambda_1)\|f\|_{p,\phi}\|g\|_{q,\psi}, \quad (1.3)$$

where the constant factor $k(\lambda_1)$ is the best possible. Moreover, if $k_\lambda(x, y)$ is finite and $k_\lambda(x, y)x^{\lambda_1-1}(k_\lambda(x, y)y^{\lambda_2-1})$ is decreasing with respect to $x > 0$ ($y > 0$), then for $a_m, b_n \geq 0$,

$$a \in l_{p,\phi} = \left\{ a; \|a\|_{p,\phi} := \left\{ \sum_{n=1}^\infty \phi(n)|a_n|^p \right\}^{\frac{1}{p}} < \infty \right\},$$

$b = \{b_n\}_{n=1}^\infty \in l_{q,\psi}$, $\|a\|_{p,\phi}, \|b\|_{q,\psi} > 0$, we have

$$\sum_{m=1}^\infty \sum_{n=1}^\infty k_\lambda(m, n)a_m b_n < k(\lambda_1)\|a\|_{p,\phi}\|b\|_{q,\psi}, \quad (1.4)$$

where, the constant factor $k(\lambda_1)$ is still the best possible. Clearly, for $\lambda = 1$, $k_1(x, y) = \frac{1}{x+y}$, $\lambda_1 = \frac{1}{q}, \lambda_2 = \frac{1}{p}$, (1.3) reduces to (1.1), while (1.4) reduces to (1.2). Some other results including multidimensional Hilbert-type inequalities and the operator expressions are provided by [7]-[22]. Some of them are extensions of (1.3) and (1.4).

On half-discrete Hilbert-type inequalities with the non-homogeneous kernels, Hardy et al. provided a few results in Theorem 351 of [1]. But they did not prove that the the constant factors are the best possible. However, Yang [23] gave a result with the kernel $\frac{1}{(1+nx)^\lambda}$ by introducing a variable and proved that the constant factor is the best possible. In 2011, Yang [24] gave a half-discrete Hardy-Hilbert's inequality with the best possible constant factor $B(\lambda_1, \lambda_2)$. Zhong *et al* ([25]–[30]) investigated several half-discrete Hilbert-type inequalities with particular kernels. Using the way of weight functions and the techniques of discrete and integral Hilbert-type inequalities with some additional conditions on the kernel, a half-discrete Hilbert-type inequality with a general homogeneous kernel of degree $-\lambda \in \mathbb{R}$ and a best constant factor $k(\lambda_1)$ is obtained as follows:

$$\int_0^\infty f(x) \sum_{n=1}^\infty k_\lambda(x, n)a_n dx < k(\lambda_1)\|f\|_{p,\phi}\|a\|_{q,\psi}, \quad (1.5)$$

which is an extension of Yang [24]'s work (see Yang and Chen [31]). A half-discrete Hilbert-type inequality with a general non-homogeneous kernel and a best constant factor is given by Yang [32].

Remark 1.1. (1) Many different kinds of Hilbert-type discrete, half-discrete and integral inequalities with applications are presented in recent twenty years. Special attention is given to new results proved during 2009-2012. Included are many generalizations, extensions and refinements of Hilbert-type discrete, half-discrete and integral inequalities involving many special functions such as beta, gamma, hypergeometric, trigonometric, hyperbolic, zeta, Bernoulli functions, Bernoulli numbers and Euler constant et al.

(2) In his five books, Yang ([3], [5], [4], [33], [34]) presented many new results on Hilbert-type operators with general homogeneous kernels of degree of real numbers and two pairs of conjugate exponents as well as the related inequalities. These research monographs contained recent developments of discrete, half-discrete and integral types of operators and inequalities with proofs, examples and applications.

In this paper, by the use of the methods of weight functions and technique of real analysis, a multidimensional more accurate half-discrete Hilbert-type inequality with the kernel of hyperbolic cotangent function is given, which is an extension of (1.5) for a particular kernel. We prove that the constant factor related to the Riemann zeta function is the best possible. The equivalent forms and the operator expressions are also considered.

2 Some Lemmas

Lemma 2.1. Suppose that $(-1)^i h^{(i)}(t) > 0 (t > 0; i = 0, 1, 2)$. Then

(i) for $b > 0, c > 1, 0 < \alpha \leq 1$, we still have

$$(-1)^i \frac{d^i}{dx^i} h((b + \ln^\alpha cx)^{\frac{1}{\alpha}}) > 0 \quad (x > 1; i = 1, 2), \quad (2.1)$$

(ii) for $\int_{\frac{1}{2}}^{\infty} h(t) dt < \infty$, we have

$$\int_1^{\infty} h(t) dt < \sum_{n=1}^{\infty} h(n) < \int_{\frac{1}{2}}^{\infty} h(t) dt. \quad (2.2)$$

Proof. (i) We find

$$\begin{aligned} \frac{d}{dx} h((b + \ln^\alpha cx)^{\frac{1}{\alpha}}) &= \frac{1}{x} h'((b + \ln^\alpha cx)^{\frac{1}{\alpha}})(b + \ln^\alpha cx)^{\frac{1}{\alpha}-1} \ln^{\alpha-1} cx < 0, \\ \frac{d^2}{dx^2} h((b + \ln^\alpha cx)^{\frac{1}{\alpha}}) &= \frac{d}{dx} \left[\frac{1}{x} h'((b + \ln^\alpha cx)^{\frac{1}{\alpha}})(b + \ln^\alpha cx)^{\frac{1}{\alpha}-1} \ln^{\alpha-1} cx \right] \\ &= -\frac{1}{x^2} h'((b + \ln^\alpha cx)^{\frac{1}{\alpha}})(b + \ln^\alpha cx)^{\frac{1}{\alpha}-1} \ln^{\alpha-1} cx \\ &\quad + \frac{1}{x^2} h''((b + \ln^\alpha cx)^{\frac{1}{\alpha}})(b + \ln^\alpha cx)^{\frac{2}{\alpha}-2} \ln^{2\alpha-2} cx \\ &\quad + \alpha \left(\frac{1}{\alpha} - 1 \right) \frac{1}{x^2} h'((b + \ln^\alpha cx)^{\frac{1}{\alpha}})(b + \ln^\alpha cx)^{\frac{1}{\alpha}-2} \ln^{2\alpha-2} cx \\ &\quad + (\alpha - 1) \frac{1}{x^2} h'((b + \ln^\alpha cx)^{\frac{1}{\alpha}})(b + \ln^\alpha cx)^{\frac{1}{\alpha}-1} \ln^{\alpha-2} cx \end{aligned}$$

$$= [-h'((b + \ln^\alpha cx)^{\frac{1}{\alpha}})(b + \ln^\alpha cx) \ln cx + h''((b + \ln^\alpha cx)^{\frac{1}{\alpha}})(b + \ln^\alpha cx)^{\frac{1}{\alpha}} \\ \times \ln^\alpha cx + b(\alpha - 1)h'((b + \ln^\alpha cx)^{\frac{1}{\alpha}})] \frac{1}{x^2} (b + \ln^\alpha cx)^{\frac{1}{\alpha}-2} \ln^{\alpha-2} cx > 0.$$

(ii) Since $h(t)$ is a decreasing convex function, by the decreasing property and Hermite-Hadamard's inequality (cf. [35]), we have

$$\int_n^{n+1} h(t) dt < h(n) < \int_{n-\frac{1}{2}}^{n+\frac{1}{2}} h(t) dt (n \in \mathbb{N}),$$

and then

$$\begin{aligned} \int_1^\infty h(t) dt &= \sum_{n=1}^\infty \int_n^{n+1} h(t) dt < \sum_{n=1}^\infty h(n) \\ &< \sum_{n=1}^\infty \int_{n-\frac{1}{2}}^{n+\frac{1}{2}} h(t) dt = \int_{\frac{1}{2}}^\infty h(t) dt. \end{aligned}$$

Hence, (2.2) follows. \square

Note. (i) If $(-1)^i h^{(i)}(t) > 0 (t > 0; i = 0, 1)$, $\int_0^\infty h(t) dt < \infty$, since

$$\int_n^{n+1} h(t) dt < h(n) < \int_{n-1}^n h(t) dt (n \in \mathbb{N}),$$

then we still have

$$\int_1^\infty h(t) dt < \sum_{n=1}^\infty h(n) < \sum_{n=1}^\infty \int_{n-1}^n h(t) dt = \int_0^\infty h(t) dt.$$

(ii) The hyperbolic cotangent function $h(t) = \coth(t) := \frac{e^t + e^{-t}}{e^t - e^{-t}}$ (cf. [36]) satisfies the condition of $(-1)^i h^{(i)}(t) > 0 (t > 0; i = 0, 1, 2)$. In fact, $h(t) = \frac{e^t + e^{-t}}{e^t - e^{-t}} > 0 (t > 0)$, we find

$$\begin{aligned} h'(t) &= \frac{e^t - e^{-t}}{e^t - e^{-t}} - \frac{(e^t + e^{-t})^2}{(e^t - e^{-t})^2} = -\frac{4}{(e^t - e^{-t})^2} < 0, \\ h''(t) &= \frac{8(e^t + e^{-t})}{(e^t - e^{-t})^3} > 0 (t > 0). \end{aligned}$$

So does $g(t) = e^{-t} \coth(t) (t > 0)$.

If $i_0, j_0 \in \mathbb{N}$ (\mathbb{N} is the set of positive integers), $\alpha, \beta > 0$, we put

$$\begin{aligned} \|x\|_\alpha &:= \left(\sum_{k=1}^{i_0} |x_k|^\alpha \right)^{\frac{1}{\alpha}} (x = (x_1, \dots, x_{i_0}) \in \mathbb{R}^{i_0}), \\ \|y\|_\beta &:= \left(\sum_{k=1}^{j_0} |y_k|^\beta \right)^{\frac{1}{\beta}} (y = (y_1, \dots, y_{j_0}) \in \mathbb{R}^{j_0}). \end{aligned} \tag{2.3}$$

Lemma 2.2. If $s \in \mathbb{N}, \gamma, M > 0$, $\Psi(u)$ is a non-negative measurable function in $(0, 1]$, and

$$D_M := \left\{ x \in \mathbb{R}_+^s; 0 < u = \sum_{i=1}^s \left(\frac{x_i}{M}\right)^\gamma \leq 1 \right\},$$

then we have

$$\int \cdots \int_{D_M} \Psi \left(\sum_{i=1}^s \left(\frac{x_i}{M}\right)^\gamma \right) dx_1 \cdots dx_s = \frac{M^s \Gamma^s(\frac{1}{\gamma})}{\gamma^s \Gamma(\frac{s}{\gamma})} \int_0^1 \Psi(u) u^{\frac{s}{\gamma}-1} du. \quad (2.4)$$

Proof. Please see Lemma 42.1 of (cf. [33]) (in P.776). \square

Applying Lemma 2.2, in view of (2.4) and the condition, it follows:

(i) For $M > 0$,

$$D_M := \left\{ x \in \mathbb{R}_+^s; 0 < u = \sum_{i=1}^s \left(\frac{x_i}{M}\right)^\gamma \leq 1 \right\},$$

we have

$$\begin{aligned} & \int \cdots \int_{\mathbb{R}_+^s} \Psi \left(\sum_{i=1}^s \left(\frac{x_i}{M}\right)^\gamma \right) dx_1 \cdots dx_s \\ &= \lim_{M \rightarrow \infty} \int \cdots \int_{D_M} \Psi \left(\sum_{i=1}^s \left(\frac{x_i}{M}\right)^\gamma \right) dx_1 \cdots dx_s \\ &= \lim_{M \rightarrow \infty} \frac{M^s \Gamma^s(\frac{1}{\gamma})}{\gamma^s \Gamma(\frac{s}{\gamma})} \int_0^1 \Psi(u) u^{\frac{s}{\gamma}-1} du; \end{aligned} \quad (2.5)$$

(ii) for $M > 1$,

$$E_M := \left\{ x \in \mathbb{R}_+^s; \frac{1}{M^\gamma} \leq u = \sum_{i=1}^s \left(\frac{x_i}{M}\right)^\gamma \leq 1 \right\},$$

we have

$$\begin{aligned} & \int \cdots \int_{\{x \in \mathbb{R}_+^s; \|x\|_\gamma \geq 1\}} \Psi \left(\sum_{i=1}^s \left(\frac{x_i}{M}\right)^\gamma \right) dx_1 \cdots dx_s \\ &= \lim_{M \rightarrow \infty} \int \cdots \int_{E_M} \Psi \left(\sum_{i=1}^s \left(\frac{x_i}{M}\right)^\gamma \right) dx_1 \cdots dx_s \\ &= \lim_{M \rightarrow \infty} \frac{M^s \Gamma^s(\frac{1}{\gamma})}{\gamma^s \Gamma(\frac{s}{\gamma})} \int_{\frac{1}{M^\gamma}}^1 \Psi(u) u^{\frac{s}{\gamma}-1} du; \end{aligned} \quad (2.6)$$

(iii) for $M > 0$,

$$F_M := \left\{ x \in \mathbb{R}_+^s; u = \sum_{i=1}^s \left(\frac{x_i}{M}\right)^\gamma \leq \frac{1}{M^\gamma} \right\},$$

we have

$$\begin{aligned}
& \int \cdots \int_{\{x \in \mathbb{R}_+^s; \|x\|_\gamma \leq 1\}} \Psi \left(\sum_{i=1}^s \left(\frac{x_i}{M} \right)^\gamma \right) dx_1 \cdots dx_s \\
&= \lim_{M \rightarrow \infty} \int \cdots \int_{F_M} \Psi \left(\sum_{i=1}^s \left(\frac{x_i}{M} \right)^\gamma \right) dx_1 \cdots dx_s \\
&= \lim_{M \rightarrow \infty} \frac{M^s \Gamma^s(\frac{1}{\gamma})}{\gamma^s \Gamma(\frac{s}{\gamma})} \int_0^{\frac{1}{M^s}} \Psi(u) u^{\frac{s}{\gamma}-1} du.
\end{aligned}$$

Note. For $\delta \in \{-1, 1\}$, $E_\delta = \{u \in (0, 1]; \frac{1}{M^\delta} \leq u^\delta\}$, in view of (ii) and (iii), it follows that

$$\begin{aligned}
& \int \cdots \int_{\{x \in \mathbb{R}_+^s; \|x\|_\gamma^\delta \geq 1\}} \Psi \left(\sum_{i=1}^s \left(\frac{x_i}{M} \right)^\gamma \right) dx_1 \cdots dx_s \\
&= \lim_{M \rightarrow \infty} \frac{M^s \Gamma^s(\frac{1}{\gamma})}{\gamma^s \Gamma(\frac{s}{\gamma})} \int_{E_\delta} \Psi(u) u^{\frac{s}{\gamma}-1} du. \tag{2.7}
\end{aligned}$$

(iv) For $M^\gamma > s$,

$$G_M := \left\{ x \in \mathbb{R}_+^s; \frac{s}{M^\gamma} < u = \sum_{i=1}^s \left(\frac{x_i}{M} \right)^\gamma \leq 1 \right\},$$

we have

$$\begin{aligned}
& \int \cdots \int_{\{x \in \mathbb{R}_+^s; x_i \geq 1\}} \Psi \left(\sum_{i=1}^s \left(\frac{x_i}{M} \right)^\gamma \right) dx_1 \cdots dx_s \\
&= \lim_{M \rightarrow \infty} \int \cdots \int_{G_M} \Psi \left(\sum_{i=1}^s \left(\frac{x_i}{M} \right)^\gamma \right) dx_1 \cdots dx_s \\
&= \lim_{M \rightarrow \infty} \frac{M^s \Gamma^s(\frac{1}{\gamma})}{\gamma^s \Gamma(\frac{s}{\gamma})} \int_{\frac{s}{M^\gamma}}^1 \Psi(u) u^{\frac{s}{\gamma}-1} du. \tag{2.8}
\end{aligned}$$

Lemma 2.3. For $s \in \mathbb{N}, \gamma > 0, \varepsilon > 0, \delta \in \{-1, 1\}$, $c = (c_1, \dots, c_s) \in [2, e]^s$, we have

$$\int \cdots \int_{\{x \in \mathbb{R}_+^s; \|x\|_\gamma^\delta \geq 1\}} \|x\|_\gamma^{-s-\delta\varepsilon} dx_1 \cdots dx_s = \frac{\Gamma^s(\frac{1}{\gamma})}{\varepsilon \gamma^{s-1} \Gamma(\frac{s}{\gamma})}, \tag{2.9}$$

$$A_s(\varepsilon) := \sum_m \frac{\|\ln cm\|_\gamma^{-s-\varepsilon}}{\prod_{i=1}^s m_i} = \frac{\Gamma^s(\frac{1}{\gamma})}{\varepsilon s^{\varepsilon/\gamma} \gamma^{s-1} \Gamma(\frac{s}{\gamma})} + O(1)(\varepsilon \rightarrow 0^+). \tag{2.10}$$

Proof. By (2.7), it follows that

$$\int_{E_\delta} u^{\frac{-\delta\varepsilon}{\gamma}-1} du = \begin{cases} \int_{M^{-\gamma}}^1 u^{\frac{-\varepsilon}{\gamma}-1} du, \delta = 1 \\ \int_0^{M^{-\gamma}} u^{\frac{\varepsilon}{\gamma}-1} du, \delta = -1 \end{cases} = \begin{cases} \frac{\gamma}{\varepsilon} (M^\varepsilon - 1), \delta = 1 \\ \frac{\gamma}{\varepsilon} M^{-\varepsilon}, \delta = -1 \end{cases},$$

$$\begin{aligned}
& \int \cdots \int_{\{x \in \mathbb{R}_+^s; \|x\|_\gamma^\delta \geq 1\}} \|x\|_\gamma^{-s-\delta\varepsilon} dx_1 \cdots dx_s \\
&= \int \cdots \int_{\{x \in \mathbb{R}_+^s; \|x\|_\gamma^\delta \geq 1\}} \left\{ M \left[\sum_{i=1}^s \left(\frac{x_i}{M} \right)^\gamma \right]^{\frac{1}{\gamma}} \right\}^{-s-\delta\varepsilon} dx_1 \cdots dx_s \\
&= \lim_{M \rightarrow \infty} \frac{M^s \Gamma^s(\frac{1}{\gamma})}{\gamma^s \Gamma(\frac{s}{\gamma})} \int_{E_\delta} (Mu^{1/\gamma})^{-s-\delta\varepsilon} u^{\frac{s}{\gamma}-1} du \\
&= \lim_{M \rightarrow \infty} \frac{M^{-\delta\varepsilon} \Gamma^s(\frac{1}{\gamma})}{\gamma^s \Gamma(\frac{s}{\gamma})} \int_{E_\delta} u^{\frac{-\delta\varepsilon}{\gamma}-1} du = \frac{\Gamma^s(\frac{1}{\gamma})}{\varepsilon \gamma^{s-1} \Gamma(\frac{s}{\gamma})}.
\end{aligned}$$

Hence we have (2.9).

For $M > s^{1/\gamma}$, we set

$$\Psi(u) = \begin{cases} 0, & 0 < u < \frac{s}{M^\gamma}, \\ (Mu^{1/\gamma})^{-s-\varepsilon}, & \frac{s}{M^\gamma} \leq u \leq 1. \end{cases}$$

Then by the decreasing property and (2.8), it follows

$$\begin{aligned}
A_s(\varepsilon) &\geq \int_{\{x \in \mathbb{R}_+^s; x_i \geq e/c_i\}} \frac{\|\ln cx\|_\gamma^{-s-\varepsilon}}{\prod_{i=1}^s x_i} dx \stackrel{u_i = \ln c_i x_i}{=} \int_{\{u \in \mathbb{R}_+^s; u_i \geq 1\}} \|u\|_\gamma^{-s-\varepsilon} du \\
&= \lim_{M \rightarrow \infty} \frac{M^s \Gamma^s(\frac{1}{\gamma})}{\gamma^s \Gamma(\frac{s}{\gamma})} \int_{s/M^\gamma}^1 (Mu^{1/\gamma})^{-s-\varepsilon} u^{\frac{s}{\gamma}-1} du = \frac{\Gamma^s(\frac{1}{\gamma})}{\varepsilon s^{\varepsilon/\gamma} \gamma^{s-1} \Gamma(\frac{s}{\gamma})}. \quad (2.11)
\end{aligned}$$

In the following, by mathematical induction, we prove that for any $s \in \mathbb{N}$,

$$A_s(\varepsilon) \leq O_s(1) + \frac{\Gamma^s(\frac{1}{\gamma})}{\varepsilon s^{\varepsilon/\gamma} \gamma^{s-1} \Gamma(\frac{s}{\gamma})} (\varepsilon \rightarrow 0^+). \quad (2.12)$$

For $s = 1$, by (2.2), it follows that

$$\begin{aligned}
A_1(\varepsilon) &= \ln^{-1-\varepsilon} c_1 + \sum_{m_1=2}^{\infty} \frac{\ln^{-1-\varepsilon} c_1 m_1}{m_1} \leq \ln^{-1-\varepsilon} c_1 + \int_{\frac{3}{2}}^{\infty} \frac{\ln^{-1-\varepsilon} c_1 x}{x} dx \\
&\leq \ln^{-1-\varepsilon} c_1 + \int_{e/c_1}^{\infty} \frac{\ln^{-1-\varepsilon} c_1 x}{x} dx = O_1(1) + \frac{1}{\varepsilon},
\end{aligned}$$

and then (2.12) is valid. Assuming that (2.12) is valid for $s-1 \in \mathbb{N}$, then for s , we set

$$A_s(\varepsilon) = \sum_{\{m \in \mathbb{N}^s; \exists i_0, m_{i_0}=1\}} \frac{\|\ln cm\|_\gamma^{-s-\varepsilon}}{\prod_{i=1}^s m_i} + \sum_{\{m \in \mathbb{N}^s; m_i \geq 2\}} \frac{\|\ln cm\|_\gamma^{-s-\varepsilon}}{\prod_{i=1}^s m_i}.$$

There exist constants $a, b \in \mathbb{R}_+$, such that

$$\sum_{\{m \in \mathbb{N}^s; \exists i_0, m_{i_0}=1\}} \frac{\|\ln cm\|_\gamma^{-s-\varepsilon}}{\prod_{i=1}^s m_i} \leq a + b \sum_{\{m \in \mathbb{N}^{s-1}; m_i \geq 1\}} \frac{\|\ln cm\|_\gamma^{-(s-1)-(1+\varepsilon)}}{\prod_{i=1}^{s-1} m_i}.$$

By the assumption of mathematical induction for $s - 1$, we find

$$\sum_{\{m \in \mathbb{N}^{s-1}; m_i \geq 1\}} \frac{\|\ln cm\|_\gamma^{-(s-1)-(1+\varepsilon)}}{\prod_{i=1}^{s-1} m_i} \leq O_{s-1}(1) + \frac{\gamma^{2-s} \Gamma^{s-1}(\frac{1}{\gamma})}{(1+\varepsilon)(s-1)^{(1+\varepsilon)/\gamma} \Gamma(\frac{s-1}{\gamma})},$$

and then

$$\sum_{\{m \in \mathbb{N}^s; \exists i_0, m_{i_0} = 1\}} \|\ln cm\|_\gamma^{-s-\varepsilon} \frac{1}{\prod_{i=1}^s m_i} \leq O_s(1).$$

By Lemma 2.1, (2.2) and (2.11), we obtain

$$\begin{aligned} \sum_{\{m \in \mathbb{N}^s; m_i \geq 2\}} \|\ln cm\|_\gamma^{-s-\varepsilon} \frac{1}{\prod_{i=1}^s m_i} &\leq \int_{\{x \in \mathbb{R}_+^s; x_i \geq \frac{3}{2}\}} \|\ln cx\|_\gamma^{-s-\varepsilon} \frac{dx}{\prod_{i=1}^s x_i} \\ &\leq \int_{\{x \in \mathbb{R}_+^s; x_i \geq e/c_i\}} \|\ln cx\|_\gamma^{-s-\varepsilon} \frac{1}{\prod_{i=1}^s x_i} dx \\ &\stackrel{u_i = \ln c_i x_i}{=} \int_{\{u \in \mathbb{R}_+^s; u_i \geq 1\}} \|u\|_\gamma^{-s-\varepsilon} du = \frac{\Gamma^s(\frac{1}{\gamma})}{\varepsilon s^{\varepsilon/\gamma} \gamma^{s-1} \Gamma(\frac{s}{\gamma})}. \end{aligned}$$

Hence we prove that (2.12) is valid for $s \in \mathbb{N}$. Therefore, we have (2.10). \square

3 Weight Functions and Equivalent Inequalities

In the following sections, we suppose that $i_0, j_0 \in \mathbb{N}$, $\alpha, \beta > 0$, $\tau = (\tau_1, \dots, \tau_{j_0}) \in [2, e]^{j_0}$, $\delta \in \{-1, 1\}$, $\sigma > 1$, $\eta > 0$, and $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$.

Definition 3.1. For $x = (x_1, \dots, x_{i_0}) \in \mathbb{R}_+^{i_0}$, $\ln \tau n = (\ln \tau_1 n_1, \dots, \ln \tau_{j_0} n_{j_0})$, $n \in \mathbb{N}^{j_0}$, define two weight functions $\omega_\delta(\sigma, n)$ and $\varpi_\delta(\sigma, x)$ as follows:

$$\omega_\delta(\sigma, n) := \|\ln \tau n\|_\beta^\sigma \int_{\mathbb{R}_+^{i_0}} e^{-\frac{\eta \|\ln \tau n\|_\beta}{\|x\|_\alpha^\delta}} \coth\left(\frac{\eta \|\ln \tau n\|_\beta}{\|x\|_\alpha^\delta}\right) \frac{dx}{\|x\|_\alpha^{i_0 + \delta\sigma}}, \quad (3.1)$$

$$\varpi_\delta(\sigma, x) := \|x\|_\alpha^{-\delta\sigma} \sum_n e^{-\frac{\eta \|\ln \tau n\|_\beta}{\|x\|_\alpha^\delta}} \coth\left(\frac{\eta \|\ln \tau n\|_\beta}{\|x\|_\alpha^\delta}\right) \frac{\prod_{j=1}^{j_0} n_j^{-1}}{\|\ln \tau n\|_\beta^{j_0 - \sigma}}, \quad (3.2)$$

where, $\sum_n = \sum_{n_{j_0}=1}^\infty \cdots \sum_{n_1=1}^\infty$.

Lemma 3.2. As the assumptions of Definition 3.1, if both $\omega_\delta(\sigma, n)$ and $\varpi_\delta(\sigma, x)$ are finite, $f(x) = f(x_1, \dots, x_{i_0}) \geq 0$, $a_n = a_{(n_1, \dots, n_{j_0})} \geq 0$, then we have the following inequality:

$$\begin{aligned} J_1 &:= \left\{ \sum_n \frac{\prod_{j=1}^{j_0} n_j^{-1} \|\ln \tau n\|_\beta^{p\sigma - j_0}}{[\omega_\delta(\sigma, n)]^{p-1}} \times \left(\int_{\mathbb{R}_+^{i_0}} e^{-\frac{\eta \|\ln \tau n\|_\beta}{\|x\|_\alpha^\delta}} \coth\left(\frac{\eta \|\ln \tau n\|_\beta}{\|x\|_\alpha^\delta}\right) f(x) dx \right)^p \right\}^{\frac{1}{p}} \\ &\leq \left\{ \int_{\mathbb{R}_+^{i_0}} \varpi_\delta(\sigma, x) \|x\|_\alpha^{p(i_0 + \delta\sigma) - i_0} f^p(x) dx \right\}^{\frac{1}{p}}, \end{aligned} \quad (3.3)$$

$$\begin{aligned}
J_2 &:= \left\{ \int_{\mathbb{R}_+^{i_0}} \frac{\|x\|_\alpha^{-\delta q \sigma - i_0}}{[\varpi_\delta(\sigma, x)]^{q-1}} \left(\sum_n e^{-\frac{\eta \|\ln \tau n\|_\beta}{\|x\|_\alpha^\delta} \coth(\frac{\eta \|\ln \tau n\|_\beta}{\|x\|_\alpha^\delta}) a_n} \right)^q dx \right\}^{\frac{1}{q}} \\
&\leq \left\{ \sum_n \omega_\delta(\sigma, n) \prod_{j=1}^{j_0} n_j^{q-1} \|\ln \tau n\|_\beta^{q(j_0-\sigma)-j_0} a_n^q \right\}^{\frac{1}{q}}. \tag{3.4}
\end{aligned}$$

Proof. By Hölder's inequality with weight (cf. [35]), it follows

$$\begin{aligned}
&\int_{\mathbb{R}_+^{i_0}} e^{-\frac{\eta \|\ln \tau n\|_\beta}{\|x\|_\alpha^\delta} \coth(\frac{\eta \|\ln \tau n\|_\beta}{\|x\|_\alpha^\delta})} f(x) dx \\
&= \int_{\mathbb{R}_+^{i_0}} e^{-\frac{\eta \|\ln \tau n\|_\beta}{\|x\|_\alpha^\delta} \coth(\frac{\eta \|\ln \tau n\|_\beta}{\|x\|_\alpha^\delta})} \left[\frac{\|x\|_\alpha^{(i_0+\delta\sigma)/q} f(x)}{\|\ln \tau n\|_\beta^{(j_0-\sigma)/p} (\prod_{j=1}^{j_0} n_j)^{1/p}} \right] \\
&\quad \times \left[\frac{\|\ln \tau n\|_\beta^{(j_0-\sigma)/p} (\prod_{j=1}^{j_0} n_j)^{1/p}}{\|x\|_\alpha^{(i_0+\delta\sigma)/q}} \right] dx \\
&\leq \left\{ \int_{\mathbb{R}_+^{i_0}} e^{-\frac{\eta \|\ln \tau n\|_\beta}{\|x\|_\alpha^\delta} \coth(\frac{\eta \|\ln \tau n\|_\beta}{\|x\|_\alpha^\delta})} \frac{\|x\|_\alpha^{(i_0+\delta\sigma)(p-1)} f^p(x)}{\|\ln \tau n\|_\beta^{j_0-\sigma} \prod_{j=1}^{j_0} n_j} dx \right\}^{\frac{1}{p}} \\
&\quad \times \left\{ \int_{\mathbb{R}_+^{i_0}} e^{-\frac{\eta \|\ln \tau n\|_\beta}{\|x\|_\alpha^\delta} \coth(\frac{\eta \|\ln \tau n\|_\beta}{\|x\|_\alpha^\delta})} \frac{\|\ln \tau n\|_\beta^{(j_0-\sigma)(q-1)}}{\|x\|_\alpha^{i_0+\delta\sigma} (\prod_{j=1}^{j_0} n_j)^{1-q}} dx \right\}^{\frac{1}{q}} \\
&= [\omega_\delta(\sigma, n)]^{\frac{1}{q}} \|\ln \tau n\|_\beta^{\frac{j_0}{p}-\sigma} \left(\prod_{j=1}^{j_0} n_j \right)^{\frac{1}{p}} \\
&\quad \times \left\{ \int_{\mathbb{R}_+^{i_0}} e^{-\frac{\eta \|\ln \tau n\|_\beta}{\|x\|_\alpha^\delta} \coth(\frac{\eta \|\ln \tau n\|_\beta}{\|x\|_\alpha^\delta})} \frac{\|x\|_\alpha^{(i_0+\delta\sigma)(p-1)} f^p(x)}{\|\ln \tau n\|_\beta^{j_0-\sigma} (\prod_{j=1}^{j_0} n_j)^{1-p}} dx \right\}^{\frac{1}{p}}. \tag{3.5}
\end{aligned}$$

Then by Lebesgue term by term integration theorem (cf. [37]), we have

$$\begin{aligned}
J_1 &\leq \left\{ \sum_n \int_{\mathbb{R}_+^{i_0}} e^{-\frac{\eta \|\ln \tau n\|_\beta}{\|x\|_\alpha^\delta} \coth(\frac{\eta \|\ln \tau n\|_\beta}{\|x\|_\alpha^\delta})} \frac{\|x\|_\alpha^{(i_0+\delta\sigma)(p-1)} f^p(x)}{\|\ln \tau n\|_\beta^{j_0-\sigma} \prod_{j=1}^{j_0} n_j} dx \right\}^{\frac{1}{p}} \\
&= \left\{ \int_{\mathbb{R}_+^{i_0}} \sum_n e^{-\frac{\eta \|\ln \tau n\|_\beta}{\|x\|_\alpha^\delta} \coth(\frac{\eta \|\ln \tau n\|_\beta}{\|x\|_\alpha^\delta})} \frac{\|x\|_\alpha^{(i_0+\delta\sigma)(p-1)} f^p(x)}{\|\ln \tau n\|_\beta^{j_0-\sigma} \prod_{j=1}^{j_0} n_j} dx \right\}^{\frac{1}{p}} \\
&= \left\{ \int_{\mathbb{R}_+^{i_0}} \varpi_\delta(\sigma, x) \|x\|_\alpha^{p(i_0+\delta\sigma)-i_0} f^p(x) dx \right\}^{\frac{1}{p}}. \tag{3.6}
\end{aligned}$$

Hence, (3.3) follows.

By Hölder's inequality with weight, we still can obtain

$$\sum_n e^{-\frac{\eta \|\ln \tau n\|_\beta}{\|x\|_\alpha^\delta} \coth(\frac{\eta \|\ln \tau n\|_\beta}{\|x\|_\alpha^\delta})} a_n \leq [\varpi_\delta(\sigma, x)]^{\frac{1}{p}} \|x\|_\alpha^{\frac{i_0}{q}+\delta\sigma}$$

$$\times \left\{ \sum_n e^{-\frac{\eta \|\ln \tau n\|_\beta}{\|x\|_\alpha^\delta}} \coth\left(\frac{\eta \|\ln \tau n\|_\beta}{\|x\|_\alpha^\delta}\right) \frac{\|\ln \tau n\|_\beta^{(j_0-\sigma)(q-1)}}{\|x\|_\alpha^{i_0+\delta\sigma}} a_n^q \right\}^{\frac{1}{q}},$$

then by Lebesgue term by term integration theorem and in the same way of obtaining (3.6), we have (3.4). \square

Lemma 3.3. *Following the assumptions of Lemma 3.2, we have the following inequality equivalent to (3.3) and (3.4):*

$$\begin{aligned} I &:= \sum_n \int_{\mathbb{R}_+^{i_0}} e^{-\frac{\eta \|\ln \tau n\|_\beta}{\|x\|_\alpha^\delta}} \coth\left(\frac{\eta \|\ln \tau n\|_\beta}{\|x\|_\alpha^\delta}\right) a_n f(x) dx \\ &\leq \left\{ \int_{\mathbb{R}_+^{i_0}} \varpi_\delta(\sigma, x) \|x\|_\alpha^{p(i_0+\delta\sigma)-i_0} f^p(x) dx \right\}^{\frac{1}{p}} \\ &\quad \times \left\{ \sum_n \omega_\delta(\sigma, n) \prod_{j=1}^{j_0} n_j^{q-1} \|\ln \tau n\|_\beta^{q(j_0-\sigma)-j_0} a_n^q \right\}^{\frac{1}{q}}. \end{aligned} \quad (3.7)$$

Proof. By Hölder's inequality (cf. [35]), it follows

$$\begin{aligned} I &= \sum_n \frac{\|\ln \tau n\|_\beta^{\frac{j_0}{q}-(j_0-\sigma)}}{[\omega_\delta(\sigma, n)]^{\frac{1}{q}} (\prod_{j=1}^{j_0} n_j)^{\frac{1}{p}}} \left[\int_{\mathbb{R}_+^{i_0}} e^{-\frac{\eta \|\ln \tau n\|_\beta}{\|x\|_\alpha^\delta}} \coth\left(\frac{\eta \|\ln \tau n\|_\beta}{\|x\|_\alpha^\delta}\right) f(x) dx \right] \\ &\quad \times \left[[\omega_\delta(\sigma, n)]^{\frac{1}{q}} (\prod_{j=1}^{j_0} n_j)^{\frac{1}{p}} \|\ln \tau n\|_\beta^{(j_0-\sigma)-\frac{j_0}{q}} a_n \right] \\ &\leq J_1 \left\{ \sum_n \omega_\delta(\sigma, n) \prod_{j=1}^{j_0} n_j^{q-1} \|\ln \tau n\|_\beta^{q(j_0-\sigma)-j_0} a_n^q \right\}^{\frac{1}{q}}. \end{aligned}$$

Then by (3.3), we have (3.7).

On the other hand, assuming that (3.7) is valid, we set

$$b_n := \frac{\|\ln \tau n\|_\beta^{p\sigma-j_0}}{[\omega_\delta(\sigma, n)]^{p-1} \prod_{j=1}^{j_0} n_j} \left(\int_{\mathbb{R}_+^{i_0}} e^{-\frac{\eta \|\ln \tau n\|_\beta}{\|x\|_\alpha^\delta}} \coth\left(\frac{\eta \|\ln \tau n\|_\beta}{\|x\|_\alpha^\delta}\right) f(x) dx \right)^{p-1}, n \in \mathbb{N}^{j_0}.$$

Then it follows

$$J_1^p = \sum_n \omega_\delta(\sigma, n) \prod_{j=1}^{j_0} n_j^{q-1} \|\ln \tau n\|_\beta^{q(j_0-\sigma)-j_0} a_n^q.$$

If $J_1 = 0$, then (3.3) is trivially valid; if $J_1 = \infty$, then by (3.6), (3.3) keeps the form of

equality ($= \infty$). Suppose that $0 < J_1 < \infty$. By (3.7), we have

$$\begin{aligned} 0 &< \sum_n \omega_\delta(\lambda_2, n) \prod_{j=1}^{j_0} n_j^{q-1} \|\ln \tau n\|_\beta^{q(j_0-\sigma)-j_0} a_n^q = J_1^p = I \\ &\leq \left\{ \int_{\mathbb{R}_+^{i_0}} \varpi_\delta(\sigma, x) \|x\|_\alpha^{p(i_0+\delta\sigma)-i_0} f^p(x) dx \right\}^{\frac{1}{p}} \\ &\quad \times \left\{ \sum_n \omega_\delta(\sigma, n) \prod_{j=1}^{j_0} n_j^{q-1} \|\ln \tau n\|_\beta^{q(j_0-\sigma)-j_0} a_n^q \right\}^{\frac{1}{q}}. \end{aligned}$$

It follows that

$$\begin{aligned} J_1 &= \left\{ \sum_n \omega_\delta(\sigma, n) \prod_{j=1}^{j_0} n_j^{q-1} \|\ln \tau n\|_\beta^{q(j_0-\sigma)-j_0} a_n^q \right\}^{\frac{1}{p}} \\ &\leq \left\{ \int_{\mathbb{R}_+^{i_0}} \varpi_\delta(\sigma, x) \|x\|_\alpha^{p(i_0+\delta\sigma)-i_0} f^p(x) dx \right\}^{\frac{1}{p}}, \end{aligned}$$

and then (3.3) follows. Hence, (3.7) and (3.3) are equivalent.

By Hölder's inequality and the same way, we can obtain

$$I \leq \left\{ \int_{\mathbb{R}_+^{i_0}} \varpi_\delta(\sigma, x) \|x\|_\alpha^{p(i_0+\delta\sigma)-i_0} f^p(x) dx \right\}^{\frac{1}{p}} J_2.$$

Then by (3.4), we have (3.7). On the other hand, assuming that (3.7) is valid, we set

$$f(x) = \frac{\|x\|_\alpha^{-q\delta\sigma-i_0}}{[\varpi_\delta(\sigma, x)]^{q-1}} \left(\sum_n e^{-\frac{\eta \|\ln \tau n\|_\beta}{\|x\|_\alpha^\delta}} \coth\left(\frac{\eta \|\ln \tau n\|_\beta}{\|x\|_\alpha^\delta}\right) a_n \right)^{q-1} (x \in \mathbb{R}_+^{i_0}).$$

Then it follows

$$J_2^q = \int_{\mathbb{R}_+^{i_0}} \varpi_\delta(\sigma, x) \|x\|_\alpha^{p(i_0+\delta\sigma)-i_0} f^p(x) dx.$$

By (3.7) and the same way, we can obtain

$$\begin{aligned} J_2 &= \left\{ \int_{\mathbb{R}_+^{i_0}} \varpi_\delta(\sigma, x) \|x\|_\alpha^{p(i_0+\delta\sigma)-i_0} f^p(x) dx \right\}^{\frac{1}{q}} \\ &\leq \left\{ \sum_n \omega_\delta(\sigma, n) \prod_{j=1}^{j_0} n_j^{q-1} \|\ln \tau n\|_\beta^{q(j_0-\sigma)-j_0} a_n^q \right\}^{\frac{1}{q}}, \end{aligned}$$

and then (3.4) is equivalent to (3.7). Hence (3.3), (3.4) and (3.7) are equivalent. \square

Lemma 3.4. *If $j_0 \in \mathbb{N} \setminus \{1\}$, $1 < \tilde{\sigma} < j_0$, $k(\tilde{\sigma}) := \int_0^\infty e^{-v} \coth(v) v^{\tilde{\sigma}-1} dv$, then we have*

$$k(\tilde{\sigma}) = \left[\left(2 - \frac{1}{2^{\tilde{\sigma}-1}} \right) \zeta(\tilde{\sigma}) - 1 \right] \Gamma(\tilde{\sigma}) \in \mathbb{R}_+, \quad (3.8)$$

$$\varpi_\lambda(\tilde{\sigma}, n) = K_2(\tilde{\sigma}) := \frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0-1}\Gamma(\frac{i_0}{\alpha})\eta^{\tilde{\sigma}}} k(\tilde{\sigma})(n \in \mathbb{N}^{j_0}), \quad (3.9)$$

where, $\zeta(\sigma) = \sum_{k=1}^{\infty} k^{-\sigma}$ ($\sigma > 1$) is the Riemann zeta function (cf. [18]).

Moreover, if $0 < \beta \leq 1$, then we have

$$K_1(\tilde{\sigma})(1 - \theta_{\tilde{\sigma}}(\|x\|_{\alpha}^{\delta})) < \omega_{\lambda}(\tilde{\sigma}, x) < K_1(\tilde{\sigma})(x \in \mathbb{R}_+^{i_0}), \quad (3.10)$$

where,

$$K_1(\tilde{\sigma}) := \frac{\Gamma^{j_0}(\frac{1}{\beta})}{\beta^{j_0-1}\Gamma(\frac{j_0}{\beta})\eta^{\tilde{\sigma}}} k(\tilde{\sigma}) \in \mathbb{R}_+, \quad (3.11)$$

$$\theta_{\tilde{\sigma}}(\|x\|_{\alpha}^{\delta}) := \frac{1}{k(\tilde{\sigma})} \int_0^{\frac{1}{\|x\|_{\alpha}^{\delta}} \eta^{1/\beta}} e^{-v} \coth(v) v^{\tilde{\sigma}-1} dv = O\left(\frac{1}{\|x\|_{\alpha}^{\delta}}\right) \in (0, 1). \quad (3.12)$$

Proof. By Lebesgue term by term integration theorem, we obtain

$$\begin{aligned} k(\tilde{\sigma}) &= \int_0^{\infty} e^{-v} \frac{e^v + e^{-v}}{e^v - e^{-v}} v^{\tilde{\sigma}-1} dv = \int_0^{\infty} \frac{e^{-v} + e^{-3v}}{1 - e^{-2v}} v^{\tilde{\sigma}-1} dv \\ &= \int_0^{\infty} (e^{-v} + e^{-3v}) \sum_{k=0}^{\infty} e^{-2kv} v^{\tilde{\sigma}-1} dv = \sum_{k=0}^{\infty} \int_0^{\infty} (e^{-(2k+1)v} + e^{-(2k+3)v}) v^{\tilde{\sigma}-1} dv \\ &= \sum_{k=0}^{\infty} \left[\frac{1}{(2k+1)^{\tilde{\sigma}}} + \frac{1}{(2k+3)^{\tilde{\sigma}}} \right] \Gamma(\tilde{\sigma}) = [2 \sum_{k=0}^{\infty} \frac{1}{(2k+1)^{\tilde{\sigma}}} - 1] \Gamma(\tilde{\sigma}) \\ &= [2(\sum_{k=1}^{\infty} \frac{1}{k^{\tilde{\sigma}}} - \frac{1}{2^{\tilde{\sigma}}} \sum_{k=1}^{\infty} \frac{1}{k^{\tilde{\sigma}}}) - 1] \Gamma(\tilde{\sigma}) = [(2 - \frac{1}{2^{\tilde{\sigma}-1}})\zeta(\tilde{\sigma}) - 1] \Gamma(\tilde{\sigma}). \end{aligned}$$

By (2.5), since $\delta = \pm 1$, we find

$$\begin{aligned} \omega_{\lambda}(\tilde{\sigma}, n) &= \|\ln \tau n\|_{\beta}^{\tilde{\sigma}} \int_{\mathbb{R}_+^{i_0}} e^{-\frac{\eta \|\ln \tau n\|_{\beta}}{M^{\delta} [\sum_{i=1}^{i_0} (\frac{x_i}{M})^{\alpha}]^{\frac{\delta}{\alpha}}}} \coth(\frac{\eta \|\ln \tau n\|_{\beta}}{M^{\delta} [\sum_{i=1}^{i_0} (\frac{x_i}{M})^{\alpha}]^{\frac{\delta}{\alpha}}}) \\ &\quad \times \frac{1}{M^{i_0+\delta\tilde{\sigma}} [\sum_{i=1}^{i_0} (\frac{x_i}{M})^{\alpha}]^{\frac{1}{\alpha}(i_0+\delta\tilde{\sigma})}} dx_1 \cdots dx_{i_0} \\ &= \lim_{M \rightarrow \infty} \frac{M^{i_0} \Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0-1} \Gamma(\frac{i_0}{\alpha})} \|\ln \tau n\|_{\beta}^{\tilde{\sigma}} \int_0^1 \frac{e^{-\frac{\eta \|\ln \tau n\|_{\beta}}{M^{\delta} u^{\frac{\delta}{\alpha}}}}}{M^{i_0+\delta\tilde{\sigma}} u^{\frac{1}{\alpha}(i_0+\delta\tilde{\sigma})}} u^{\frac{i_0}{\alpha}-1} du \\ &= \frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0-1} \Gamma(\frac{i_0}{\alpha}) \eta^{\tilde{\sigma}}} \int_0^{\infty} e^{-v} \coth(v) v^{\tilde{\sigma}-1} dv (v = \frac{\eta \|\ln \tau n\|_{\beta}}{M^{\delta} u^{\frac{\delta}{\alpha}}}). \end{aligned}$$

Hence, we have (3.8) and (3.9).

Moreover, by Lemma 2.1 and (2.5), we obtain

$$\begin{aligned}
\varpi_\delta(\tilde{\sigma}, x) &< \int_{\{y \in \mathbb{R}_+^{j_0}; y_i \geq \frac{1}{2}\}} e^{-\frac{\eta \|\ln \tau y\|_\beta}{\|x\|_\alpha^\delta}} \coth\left(\frac{\eta \|\ln \tau y\|_\beta}{\|x\|_\alpha^\delta}\right) \frac{\|x\|_\alpha^{-\delta\tilde{\sigma}} dy}{\|\ln \tau y\|_\beta^{j_0-\tilde{\sigma}} \prod_{j=1}^{j_0} y_j} \\
&= \|x\|_\alpha^{-\delta\tilde{\sigma}} \int_{\{u \in \mathbb{R}_+^{j_0}; u_i \geq \frac{1}{2} \tau_i\}} e^{-\frac{\eta \|u\|_\beta}{\|x\|_\alpha^\delta}} \coth\left(\frac{\eta \|u\|_\beta}{\|x\|_\alpha^\delta}\right) \frac{du}{\|u\|_\beta^{j_0-\tilde{\sigma}}} (u_i = \ln \tau_i y_i) \\
&\leq \|x\|_\alpha^{-\delta\tilde{\sigma}} \int_{\mathbb{R}_+^{j_0}} e^{-\frac{\eta \|u\|_\beta}{\|x\|_\alpha^\delta}} \coth\left(\frac{\eta \|u\|_\beta}{\|x\|_\alpha^\delta}\right) \frac{du}{\|u\|_\beta^{j_0-\tilde{\sigma}}} \\
&= \int_{\mathbb{R}_+^{j_0}} e^{-\left(\frac{\eta M [\sum_{j=1}^{j_0} (\frac{u_j}{M})^\beta]^{\frac{1}{\beta}}}{\|x\|_\alpha^\delta}\right)} \coth\left(\frac{\eta M [\sum_{j=1}^{j_0} (\frac{u_j}{M})^\beta]^{\frac{1}{\beta}}}{\|x\|_\alpha^\delta}\right) \frac{M^{\tilde{\sigma}-j_0} \|x\|_\alpha^{-\delta\tilde{\sigma}} du}{[\sum_{j=1}^{j_0} (\frac{u_j}{M})^\beta]^{\frac{j_0-\tilde{\sigma}}{\beta}}} \\
&= \lim_{M \rightarrow \infty} \frac{M^{j_0} \Gamma^{j_0}(\frac{1}{\beta})}{\beta^{j_0} \Gamma(\frac{j_0}{\beta})} \|x\|_\alpha^{-\delta\tilde{\sigma}} \int_0^1 e^{-\frac{\eta M t^{\frac{1}{\beta}}}{\|x\|_\alpha^\delta}} \coth\left(\frac{\eta M t^{\frac{1}{\beta}}}{\|x\|_\alpha^\delta}\right) \frac{t^{\frac{j_0}{\beta}-1} dt}{M^{j_0-\tilde{\sigma}} t^{\frac{j_0-\tilde{\sigma}}{\beta}}} \\
&= \frac{\Gamma^{j_0}(\frac{1}{\beta})}{\beta^{j_0-1} \Gamma(\frac{j_0}{\beta}) \eta^{\tilde{\sigma}}} \int_0^\infty e^{-v} \coth(v) v^{\tilde{\sigma}-1} dv = K_2(\tilde{\sigma})(v = \frac{\eta M t^{\frac{1}{\beta}}}{\|x\|_\alpha^\delta}).
\end{aligned}$$

By Lemma 2.1, (2.8) and (3.8), we find

$$\begin{aligned}
\varpi_\lambda(\tilde{\sigma}, x) &> \int_{\{y \in \mathbb{R}_+^{j_0}; y_i \geq e/\tau_i\}} e^{-\frac{\eta \|\ln \tau y\|_\beta}{\|x\|_\alpha^\delta}} \coth\left(\frac{\eta \|\ln \tau y\|_\beta}{\|x\|_\alpha^\delta}\right) \frac{\|x\|_\alpha^{-\delta\tilde{\sigma}} dy}{\|\ln \tau y\|_\beta^{j_0-\tilde{\sigma}} \prod_{j=1}^{j_0} y_j} \\
&= \|x\|_\alpha^{-\delta\tilde{\sigma}} \int_{\{u \in \mathbb{R}_+^{j_0}; u_i \geq 1\}} e^{-\frac{\eta \|u\|_\beta}{\|x\|_\alpha^\delta}} \coth\left(\frac{\eta \|u\|_\beta}{\|x\|_\alpha^\delta}\right) \frac{du}{\|u\|_\beta^{j_0-\tilde{\sigma}}} (u_i = \ln \tau_i y_i) \\
&= \|x\|_\alpha^{-\delta\tilde{\sigma}} \lim_{M \rightarrow \infty} \frac{M^{j_0} \Gamma^{j_0}(\frac{1}{\beta})}{\beta^{j_0-1} \Gamma(\frac{j_0}{\beta})} \int_{\frac{1}{M^\beta}}^1 e^{-\frac{\eta M t^{\frac{1}{\beta}}}{\|x\|_\alpha^\delta}} \coth\left(\frac{\eta M t^{\frac{1}{\beta}}}{\|x\|_\alpha^\delta}\right) \frac{t^{\frac{j_0}{\beta}-1} dt}{M^{j_0-\tilde{\sigma}} t^{\frac{j_0-\tilde{\sigma}}{\beta}}} \\
&= \frac{\Gamma^{j_0}(\frac{1}{\beta})}{\beta^{j_0-1} \Gamma(\frac{j_0}{\beta}) \eta^{\tilde{\sigma}}} \int_{\frac{1}{\|x\|_\alpha^\delta} \eta^{1/\beta}}^\infty e^{-v} \coth(v) v^{\tilde{\sigma}-1} dv (v = \frac{\eta \|u\|_\beta}{\|x\|_\alpha^\delta}) \\
&= \frac{\Gamma^{j_0}(\frac{1}{\beta})}{\beta^{j_0-1} \Gamma(\frac{j_0}{\beta}) \eta^{\tilde{\sigma}}} k(\tilde{\sigma}) [1 - \theta_{\tilde{\sigma}}(\|x\|_\alpha^\delta)] > 0, \\
0 &< \theta_{\tilde{\sigma}}(\|x\|_\alpha^\delta) = \frac{1}{k(\tilde{\sigma})} \int_0^{\frac{1}{\|x\|_\alpha^\delta} \eta^{1/\beta}} e^{-v} \coth(v) v^{\tilde{\sigma}-1} dv.
\end{aligned}$$

Since $e^{-v} \coth(v) v^{\tilde{\sigma}-1} \rightarrow 0$ ($v \rightarrow 0^+$, or $v \rightarrow \infty$), there exists a positive constant L , such that $e^{-v} \coth(v) v^{\tilde{\sigma}-1} \leq L$ ($v \in (0, \infty)$), and then

$$\theta_{\tilde{\sigma}}(\|x\|_\alpha^\delta) \leq \frac{L}{k(\tilde{\sigma})} \int_0^{\frac{1}{\|x\|_\alpha^\delta} \eta^{1/\beta}} dv = \frac{L}{k(\tilde{\sigma})} \frac{1}{\|x\|_\alpha^\delta} \eta^{1/\beta}.$$

Hence we have (3.10), (3.11) and (3.12). \square

Note. The following references [38]-[43] provide an extensive theory and applications of Analysis number theory related to the Riemann zeta function that offers a source of study for further research on Hilbert-type inequalities.

4 Main Results and Operator Expressions

We set $\Phi_\delta(x) := \|x\|_\alpha^{p(i_0 + \delta\sigma) - i_0}$ ($x \in \mathbb{R}_+^{i_0}$),

$$\Psi(n) := \prod_{j=1}^{j_0} n_j^{q-1} \|\ln \tau n\|_\beta^{q(j_0 - \sigma) - j_0} (n \in \mathbb{N}^{j_0}),$$

wherfrom, $[\Phi_\delta(x)]^{1-q} = \|x\|_\alpha^{-q\delta\sigma - i_0}$, $[\Psi(n)]^{1-p} = \prod_{j=1}^{j_0} n_j^{-1} \|\ln \tau n\|_\beta^{p\sigma - j_0}$.

Theorem 4.1. If $0 < \beta \leq 1$, $j_0 \in \mathbb{N} \setminus \{1\}$, $f(x) = f(x_1, \dots, x_{i_0}) \geq 0$, $a_n = a_{(n_1, \dots, n_{j_0})} \geq 0$,

$$\begin{aligned} 0 &< \|f\|_{p, \Phi_\delta} = \left\{ \int_{\mathbb{R}_+^{i_0}} \Phi_\delta(x) f^p(x) dx \right\}^{\frac{1}{p}} < \infty, \\ 0 &< \|a\|_{q, \Psi} = \left\{ \sum_n \Psi(n) a_n^q \right\}^{\frac{1}{q}} < \infty, \end{aligned}$$

then we have the following equivalent inequalities with the best possible constant factor $K(\sigma)$:

$$I = \sum_n \int_{\mathbb{R}_+^{i_0}} e^{-\frac{\eta \|\ln \tau n\|_\beta}{\|x\|_\alpha^\delta}} \coth\left(\frac{\eta \|\ln \tau n\|_\beta}{\|x\|_\alpha^\delta}\right) a_n f(x) dx < K(\sigma) \|f\|_{p, \Phi_\delta} \|a\|_{q, \Psi}, \quad (4.1)$$

$$\begin{aligned} J &:= \left\{ \sum_n \frac{\prod_{j=1}^{j_0} n_j^{-1}}{\|\ln \tau n\|_\beta^{j_0 - p\sigma}} \left(\int_{\mathbb{R}_+^{i_0}} e^{-\frac{\eta \|\ln \tau n\|_\beta}{\|x\|_\alpha^\delta}} \coth\left(\frac{\eta \|\ln \tau n\|_\beta}{\|x\|_\alpha^\delta}\right) f(x) dx \right)^p \right\}^{\frac{1}{p}} \\ &< K(\sigma) \|f\|_{p, \Phi_\delta}, \end{aligned} \quad (4.2)$$

$$\begin{aligned} H &:= \left\{ \int_{\mathbb{R}_+^{i_0}} \frac{1}{\|x\|_\alpha^{i_0 + q\delta\sigma}} \left(\sum_n e^{-\frac{\eta \|\ln \tau n\|_\beta}{\|x\|_\alpha^\delta}} \coth\left(\frac{\eta \|\ln \tau n\|_\beta}{\|x\|_\alpha^\delta}\right) a_n \right)^q dx \right\}^{\frac{1}{q}} \\ &< K(\sigma) \|a\|_{q, \Psi}, \end{aligned} \quad (4.3)$$

where, $k(\sigma) = [(2 - \frac{1}{2^{\sigma-1}})\zeta(\sigma) - 1]\Gamma(\sigma)$,

$$K(\sigma) = \left[\frac{\Gamma^{j_0}(\frac{1}{\beta})}{\beta^{j_0-1} \Gamma(\frac{j_0}{\beta})} \right]^{\frac{1}{p}} \left[\frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0-1} \Gamma(\frac{i_0}{\alpha})} \right]^{\frac{1}{q}} \frac{k(\sigma)}{\eta^\sigma}.$$

Proof. By Lemma 3.2, Lemma 3.3 and Lemma 3.4, we have equivalent inequalities (4.1), (4.2) and (4.3). By Hölder's inequality, we still have

$$I \leq J \left\{ \sum_n \prod_{j=1}^{j_0} n_j^{q-1} \|\ln \tau n\|_{\beta}^{q(j_0-\sigma)-j_0} a_n^q \right\}^{\frac{1}{q}}, \quad (4.4)$$

$$I \leq \left\{ \int_{\mathbb{R}_+^{i_0}} \|x\|_{\alpha}^{p(i_0+\delta\sigma)-i_0} f^p(x) dx \right\}^{\frac{1}{p}} H. \quad (4.5)$$

For $0 < \varepsilon < q(\sigma - 1)$, we set $\tilde{f}(x), \tilde{a}_n$ as follows:

$$\tilde{f}(x) := \begin{cases} 0, & 0 < \|x\|_{\alpha}^{\delta} < 1, \\ \|x\|_{\alpha}^{-\delta\sigma - \frac{\delta\varepsilon}{p} - i_0}, & \|x\|_{\alpha}^{\delta} \geq 1, \end{cases}$$

$$\tilde{a}_n := \|\ln \tau n\|_{\beta}^{(\sigma - \frac{\varepsilon}{q}) - j_0} \frac{1}{\prod_{j=1}^{j_0} n_j}, \quad n \in \mathbb{N}^{j_0}.$$

Then for $\tilde{\sigma} = \sigma - \frac{\varepsilon}{q} (> 1)$, in view of (2.9), (2.10) and (3.10), we find

$$\begin{aligned} \|\tilde{f}\|_{p, \Phi_{\delta}} \|\tilde{a}\|_{q, \Psi} &= \left\{ \int_{\{x \in \mathbb{R}_+^{i_0}; \|x\|_{\alpha}^{\delta} \geq 1\}} \|x\|_{\alpha}^{-i_0 - \delta\varepsilon} dx \right\}^{\frac{1}{p}} \left\{ \sum_n \frac{\|\ln \tau n\|_{\beta}^{-j_0 - \varepsilon}}{\prod_{j=1}^{j_0} n_j} \right\}^{\frac{1}{q}} \\ &= \frac{1}{\varepsilon} \left\{ \frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0-1} \Gamma(\frac{i_0}{\alpha})} \right\}^{\frac{1}{p}} \left\{ \frac{\Gamma^{j_0}(\frac{1}{\beta})}{j_0^{\varepsilon/\beta} \beta^{j_0-1} \Gamma(\frac{j_0}{\beta})} + \varepsilon O(1) \right\}^{\frac{1}{q}}, \end{aligned}$$

$$\begin{aligned} \tilde{I} &:= \int_{\mathbb{R}_+^{i_0}} \sum_n e^{-\frac{\eta \|\ln \tau n\|_{\beta}}{\|x\|_{\alpha}^{\delta}}} \coth\left(\frac{\eta \|\ln \tau n\|_{\beta}}{\|x\|_{\alpha}^{\delta}}\right) \tilde{a}_n \tilde{f}(x) dx \\ &= \int_{\{x \in \mathbb{R}_+^{i_0}; \|x\|_{\alpha}^{\delta} \geq 1\}} \|x\|_{\alpha}^{-i_0 - \delta\varepsilon} \varpi_{\delta}(\tilde{\sigma}, x) dx \\ &\geq K_1(\tilde{\sigma}) \int_{\{x \in \mathbb{R}_+^{i_0}; \|x\|_{\alpha}^{\delta} \geq 1\}} \|x\|_{\alpha}^{-i_0 - \delta\varepsilon} (1 - O(\frac{1}{\|x\|_{\alpha}^{\delta}})) dx \\ &= \frac{1}{\varepsilon} K_1(\tilde{\sigma}) \left[\frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0-1} \Gamma(\frac{i_0}{\alpha})} - \varepsilon O_{\tilde{\sigma}}(1) \right]. \end{aligned}$$

If there exists a constant $K \leq K(\sigma)$, such that (4.1) is valid when replacing $K(\sigma)$ by K , then in particular, we have

$$\begin{aligned} K_1(\tilde{\sigma}) \left[\frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0-1} \Gamma(\frac{i_0}{\alpha})} - \varepsilon O_{\tilde{\sigma}}(1) \right] &\leq \varepsilon \tilde{I} < \varepsilon K \|\tilde{f}\|_{p, \Phi_{\delta}} \|\tilde{a}\|_{q, \Psi} \\ &= K \left\{ \frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0-1} \Gamma(\frac{i_0}{\alpha})} \right\}^{\frac{1}{p}} \left\{ \frac{\Gamma^{j_0}(\frac{1}{\beta})}{j_0^{\varepsilon/\beta} \beta^{j_0-1} \Gamma(\frac{j_0}{\beta})} + \varepsilon O(1) \right\}^{\frac{1}{q}}, \end{aligned}$$

and then $K(\sigma) \leq K(\epsilon \rightarrow 0^+)$. Hence $K = K(\sigma)$ is the best possible constant factor of (4.1).

By the equivalency, we can prove that the constant factor $K(\sigma)$ in (4.2) and (4.3) is the best possible. Otherwise, we would reach a contradiction by (4.4) and (4.5) that the constant factor $K(\sigma)$ in (4.1) is not the best possible. \square

We define two real weight normal spaces $L_{p,\Phi_\delta}(\mathbb{R}_+^{i_0})$ and $l_{q,\Psi}$ as follows:

$$\begin{aligned} L_{p,\Phi_\delta}(\mathbb{R}_+^{i_0}) &:= \left\{ f; \|f\|_{p,\Phi_\delta} = \left\{ \int_{\mathbb{R}_+^{i_0}} \Phi_\delta(x) |f(x)|^p dx \right\}^{\frac{1}{p}} < \infty \right\}, \\ l_{q,\Psi} &:= \left\{ a = \{a_n\}; \|a\|_{q,\Psi} = \left\{ \sum_n \Psi(n) |a_n|^q \right\}^{\frac{1}{q}} < \infty \right\}. \end{aligned}$$

Following the assumptions of Theorem 4.1, in view of

$$J < K(\sigma) \|f\|_{p,\Phi_\delta}, H < K(\sigma) \|a\|_{q,\Psi},$$

we can give the following definition:

Definition 4.2. Define a first kind of multidimensional half-discrete Hilbert-type operator

$$T_1 : L_{p,\Phi_\delta}(\mathbb{R}_+^{i_0}) \rightarrow l_{p,\Psi^{1-p}}$$

as follows: For $f \in L_{p,\Phi_\delta}(\mathbb{R}_+^{i_0})$, there exists an unique representation $T_1 f \in l_{p,\Psi^{1-p}}$, satisfying

$$(T_1 f)(n) := \int_{\mathbb{R}_+^{i_0}} e^{-\frac{\eta \|\ln \tau n\|_\beta}{\|x\|_\alpha^\delta}} \coth\left(\frac{\eta \|\ln \tau n\|_\beta}{\|x\|_\alpha^\delta}\right) f(x) dx (n \in \mathbb{N}^{i_0}). \quad (4.6)$$

For $a \in l_{q,\Psi}$, we define the following formal inner product of $T_1 f$ and a as follows:

$$(T_1 f, a) := \sum_n a_n \int_{\mathbb{R}_+^{i_0}} e^{-\frac{\eta \|\ln \tau n\|_\beta}{\|x\|_\alpha^\delta}} \coth\left(\frac{\eta \|\ln \tau n\|_\beta}{\|x\|_\alpha^\delta}\right) f(x) dx. \quad (4.7)$$

Define a second kind of multidimensional half-discrete Hilbert-type operator

$$T_2 : l_{q,\Psi} \rightarrow L_{q,\Phi_\delta^{1-q}}(\mathbb{R}_+^{i_0})$$

as follows: For $a \in l_{q,\Psi}$, there exists an unique representation $T_2 a \in L_{q,\Phi_\delta^{1-q}}(\mathbb{R}_+^{i_0})$, satisfying

$$(T_2 a)(x) := \sum_n e^{-\frac{\eta \|\ln \tau n\|_\beta}{\|x\|_\alpha^\delta}} \coth\left(\frac{\eta \|\ln \tau n\|_\beta}{\|x\|_\alpha^\delta}\right) a_n (x \in \mathbb{R}_+^{i_0}). \quad (4.8)$$

For $f \in L_{p,\Phi_\delta}(\mathbb{R}_+^{i_0})$, we define the following formal inner product of f and $T_2 a$ as follows:

$$(f, T_2 a) := \int_{\mathbb{R}_+^{i_0}} \sum_n e^{-\frac{\eta \|\ln \tau n\|_\beta}{\|x\|_\alpha^\delta}} \coth\left(\frac{\eta \|\ln \tau n\|_\beta}{\|x\|_\alpha^\delta}\right) a_n f(x) dx. \quad (4.9)$$

Then by Theorem 4.1, for $0 < \|f\|_{p,\Phi_\delta}, \|a\|_{q,\Psi} < \infty$, we have the following equivalent inequalities:

$$(T_1 f, a) = (f, T_2 a) < K(\sigma) \|f\|_{p,\Phi_\delta} \|a\|_{q,\Psi}, \quad (4.10)$$

$$\|T_1 f\|_{p,\Psi^{1-p}} < K(\sigma) \|f\|_{p,\Phi_\delta}, \quad (4.11)$$

$$\|T_2 a\|_{q,\Phi_\delta^{1-q}} < K(\sigma) \|a\|_{q,\Psi}. \quad (4.12)$$

It follows that T_1 and T_2 are bounded since

$$\begin{aligned} \|T_1\| &:= \sup_{f(\neq 0) \in L_{p,\Phi_\delta}(\mathbb{R}_+^{i_0})} \frac{\|T_1 f\|_{p,\Psi^{1-p}}}{\|f\|_{p,\Phi_\delta}} \leq K(\sigma), \\ \|T_2\| &:= \sup_{a(\neq 0) \in l_{q,\Psi}} \frac{\|T_2 a\|_{q,\Phi_\delta^{1-q}}}{\|a\|_{q,\Psi}} \leq K(\sigma). \end{aligned}$$

Since the constant factor $K(\sigma)$ in (4.11) and (4.12) is the best possible, we have

$$\|T_1\| = \|T_2\| = K(\sigma) = \left[\frac{\Gamma^{j_0}(\frac{1}{\beta})}{\beta^{j_0-1} \Gamma(\frac{j_0}{\beta})} \right]^{\frac{1}{p}} \left[\frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0-1} \Gamma(\frac{i_0}{\alpha})} \right]^{\frac{1}{q}} \frac{k(\sigma)}{\eta^\sigma}. \quad (4.13)$$

Remark 4.3. For $\tau = e$ in (4.1), we have the following inequality:

$$\sum_n \int_{\mathbb{R}_+^{i_0}} e^{-\frac{\eta \|\ln en\|_\beta}{\|x\|_\alpha^\delta}} \coth\left(\frac{\eta \|\ln en\|_\beta}{\|x\|_\alpha^\delta}\right) a_n f(x) dx < K(\sigma) \|f\|_{p,\Phi_\delta} \|a\|_{q,\Psi}. \quad (4.14)$$

Hence, (4.1) is a more accurate inequality of (4.14). In particular, for $\delta = -1$, we have the following inequality with the non-homogeneous kernel:

$$\begin{aligned} &\sum_n \int_{\mathbb{R}_+^{i_0}} e^{-\eta \|x\|_\alpha \|\ln en\|_\beta} \coth(\eta \|x\|_\alpha \|\ln en\|_\beta) a_n f(x) dx \\ &< K(\sigma) \|f\|_{p,\Phi_{-1}} \|a\|_{q,\Psi}. \end{aligned} \quad (4.15)$$

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