A Multidimensional Hilbert-Type Integral Inequality Related to the Riemann Zeta Function

Michael Th. Rassias and Bicheng Yang

Abstract In this chapter, using methods of weight functions and techniques of real analysis, we provide a multidimensional Hilbert-type integral inequality with a homogeneous kernel of degree 0 as well as a best possible constant factor related to the Riemann zeta function. Some equivalent representations and certain reverses are obtained. Furthermore, we also consider operator expressions with the norm and some particular results.
Key words Hilbert-type integral inequality; Hilbert-type integral operator; Riemann zeta function; Gamma function; weight function;
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1 Introduction

If $p > 1, \frac{1}{p} + \frac{1}{q} = 1, f(x), g(y) \geq 0, f \in L^p(\mathbf{R}_+), g \in L^q(\mathbf{R}_+)$,

$$||f||_p = \left\{ \int_0^\infty f^p(x)dx \right\}^{\frac{1}{p}} > 0,$$

$||g||_q > 0$, then we have the following Hardy-Hilbert's integral inequality (cf. [1]):

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} dxdy < \frac{\pi}{\sin(\pi/p)} ||f||_p ||g||_q, \tag{1}$$

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where the constant factor $\frac{\pi}{\sin(\pi/p)}$ is the best possible.
If $a_m, b_n \geq 0$, $a = \{a_m\}_{m=1}^{\infty} \in l^p$, $b = \{b_n\}_{n=1}^{\infty} \in l^q$, where

$$||a||_p = \{\sum_{m=1}^{\infty} a_m^p\}^{\frac{1}{p}} > 0, ||b||_q > 0,$$

then we still have the following discrete variant of the above inequality with the same best constant $\frac{\pi}{\sin(\pi/p)}$, that is

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{m+n} < \frac{\pi}{\sin(\pi/p)}||a||_p||b||_q. \tag{2}$$

Inequalities (1) and (2) are important in mathematical analysis and its applications (cf. [1], [2], [3], [4], [5], [6], [7]).

In 1998, by introducing an independent parameter $\lambda \in (0,1]$, Yang [8] presented an extension of (1) for $p = q = 2$. In 2009 and 2011, Yang [4], [5] provided some extensions of (1) and (2) as follows:
If $\lambda_1, \lambda_2, \lambda \in \mathbf{R}, \lambda_1 + \lambda_2 = \lambda, k_\lambda(x,y)$ is a non-negative homogeneous function of degree $-\lambda$, with

$$k(\lambda_1) = \int_0^\infty k_\lambda(t,1) t^{\lambda_1 - 1} dt \in \mathbf{R}_+,$$

and

$$\phi(x) = x^{p(1-\lambda_1)-1}, \psi(y) = y^{q(1-\lambda_2)-1}, f(x), g(y) \geq 0,$$

$$f \in L_{p,\phi}(\mathbf{R}_+) = \left\{ f : ||f||_{p,\phi} := \left\{ \int_0^\infty \phi(x)|f(x)|^p dx \right\}^{\frac{1}{p}} < \infty \right\},$$

$$g \in L_{q,\psi}(\mathbf{R}_+), ||f||_{p,\phi}, ||g||_{q,\psi} > 0,$$

then we have

$$\int_0^\infty \int_0^\infty k_\lambda(x,y) f(x) g(y) dx dy < k(\lambda_1)||f||_{p,\phi}||g||_{q,\psi}, \tag{3}$$

where the constant factor $k(\lambda_1)$ is the best possible. Moreover, if $k_\lambda(x,y)$ is finite and

$$k_\lambda(x,y) x^{\lambda_1 - 1} (k_\lambda(x,y) y^{\lambda_2 - 1})$$

is decreasing with respect to $x > 0$ $(y > 0)$, then for $a_m, b_n \geq 0$,

$$a \in l_{p,\phi} = \left\{ a : ||a||_{p,\phi} := \left\{ \sum_{n=1}^{\infty} \phi(n)|a_n|^p \right\}^{\frac{1}{p}} < \infty \right\},$$

$$b = \{b_n\}_{n=1}^{\infty} \in l_{q,\psi},\ ||a||_{p,\phi}, ||b||_{q,\psi} > 0,$$

we have

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} k_\lambda(m,n)a_m b_n < k(\lambda_1)||a||_{p,\phi}||b||_{q,\psi}, \tag{4}$$

where, the constant factor $k(\lambda_1)$ is still the best possible.

Clearly, for

$$\lambda = 1, k_1(x,y) = \frac{1}{x+y},\ \lambda_1 = \frac{1}{q}, \lambda_2 = \frac{1}{p},$$

(3) reduces to (1), while (4) reduces to (2). Some further results including a few multidimensional Hilbert-type integral inequalities are provided in [9] – [19].

In this chapter, using methods of weight functions and techniques of real analysis, we present a new multidimensional Hilbert-type integral inequality with a homogeneous kernel of degree 0 as well as a best possible constant factor related to the Riemann zeta function and the Gamma function, which is an extension of the double case as follows:

$$\int_0^\infty \int_0^\infty \left(\coth(\frac{x}{y}) - 1\right) f(x)g(y)dxdy < \frac{\Gamma(\sigma)}{2^{\sigma-1}}\zeta(\sigma)||f||_{p,\varphi}||g||_{q,\psi}, \tag{5}$$

where, $\zeta(\cdot)$ is the Riemann zeta function and $\Gamma(\cdot)$ is the Gamma function (cf. [20], [22]). Some equivalent forms and reverses are obtained. Furthermore, we also consider the operator expressions with the norm and certain particular results. For a number of fundamental properties of the Riemann zeta function and the Gamma function, especially in Analytic Number Theory and related subjects, the reader is referred to [21] – [27], [31].

2 Some Lemmas

If $m, n \in \mathbf{N}$ ($\mathbf{N}$ is the set of positive integers), $\alpha, \beta > 0$, we define

$$||x||_\alpha := \left(\sum_{k=1}^{m} |x_k|^\alpha\right)^{\frac{1}{\alpha}} \quad (x = (x_1, \cdots, x_m) \in \mathbf{R}^m),$$

$$||y||_\beta := \left(\sum_{k=1}^{n} |y_k|^\beta\right)^{\frac{1}{\beta}} \quad (y = (y_1, \cdots, y_n) \in \mathbf{R}^n).$$

Lemma 1 *If $s \in \mathbf{N}, \gamma, M > 0, \Psi(u)$ is a non negative measurable function defined in $(0,1]$, and*

$$D_M^s := \left\{ x \in \mathbf{R}_+^s : 0 < u = \sum_{i=1}^{s} \left(\frac{x_i}{M}\right)^\gamma \leq 1 \right\},$$

then we have (cf. [7])

$$\int \cdots \int_{D_M^s} \Psi\left(\sum_{i=1}^{s}\left(\frac{x_i}{M}\right)^{\gamma}\right) dx_1 \cdots dx_s = \frac{M^s \Gamma^s(\frac{1}{\gamma})}{\gamma^s \Gamma(\frac{s}{\gamma})} \int_0^1 \Psi(u) u^{\frac{s}{\gamma}-1} du. \tag{6}$$

Lemma 2 (See [18]) *If* $s \in \mathbf{N}, \gamma > 0$, *and* $\varepsilon \geq 0$, *then*

$$\int \cdots \int_{\{x \in \mathbf{R}_+^s : ||x||_\gamma \geq 1\}} ||x||_\gamma^{-s-\varepsilon} dx_1 \cdots dx_s = \begin{cases} \frac{\Gamma^s(\frac{1}{\gamma})}{\varepsilon \gamma^{s-1} \Gamma(\frac{s}{\gamma})}, & \varepsilon > 0 \\ \infty, & \varepsilon = 0 \end{cases}. \tag{7}$$

Definition 1 *For* $x = (x_1, \cdots, x_m) \in \mathbf{R}_+^m, y = (y_1, \cdots, y_n) \in \mathbf{R}_+^n$, $\sigma > 1$, *we define two weight functions* $\omega(\sigma, y)$ *and* $\varpi(\sigma, x)$, *as follows*

$$\omega(\sigma, y) := ||y||_\beta^{-\sigma} \int_{\mathbf{R}_+^m} \left(\coth \frac{||x||_\alpha}{||y||_\beta} - 1\right) \frac{dx}{||x||_\alpha^{m-\sigma}}, \tag{8}$$

$$\varpi(\sigma, x) := ||x||_\alpha^{\sigma} \int_{\mathbf{R}_+^n} \left(\coth \frac{||x||_\alpha}{||y||_\beta} - 1\right) \frac{dy}{||y||_\beta^{n+\sigma}}, \tag{9}$$

where $\coth u = (e^u + e^{-u})/(e^u - e^{-u})$ *is the hyperbolic cotangent function (cf. [28]).*

By (6), setting

$$v = \frac{M u^{\frac{1}{\alpha}}}{||y||_\beta},$$

we find

$$\begin{aligned} \omega(\sigma, y) &= ||y||_\beta^{-\sigma} \lim_{M \to \infty} \int_{D_M^m} \left(\coth \frac{||x||_\alpha}{||y||_\beta} - 1\right) \frac{dx}{||x||_\alpha^{m-\sigma}} \\ &= ||y||_\beta^{-\sigma} \lim_{M \to \infty} \int_{D_M^m} \frac{\coth \frac{M}{||y||_\beta} [\sum_{i=1}^m (\frac{x_i}{M})^\alpha]^{\frac{1}{\alpha}} - 1}{M^{m-\sigma} [\sum_{i=1}^m (\frac{x_i}{M})^\alpha]^{\frac{m-\sigma}{\alpha}}} dx \\ &= ||y||_\beta^{-\sigma} \lim_{M \to \infty} \frac{M^m \Gamma^m(\frac{1}{\alpha})}{\alpha^m \Gamma(\frac{m}{\alpha})} \int_0^1 \frac{\coth(\frac{M}{||y||_\beta}) u^{\frac{1}{\alpha}} - 1}{M^{m-\sigma} u^{\frac{m-\sigma}{\alpha}}} u^{\frac{m}{\alpha}-1} du \\ &= ||y||_\beta^{-\sigma} \lim_{M \to \infty} \frac{M^\sigma \Gamma^m(\frac{1}{\alpha})}{\alpha^m \Gamma(\frac{m}{\alpha})} \int_0^1 \left(\coth(\frac{M}{||y||_\beta}) u^{\frac{1}{\alpha}} - 1\right) u^{\frac{\sigma}{\alpha}-1} du \\ &= \frac{\Gamma^m(\frac{1}{\alpha})}{\alpha^{m-1} \Gamma(\frac{m}{\alpha})} \int_0^\infty (\coth v - 1) v^{\sigma-1} dv, \end{aligned}$$

and in view of the Lebesgue term by term theorem (cf. [29]), it follows

$$\begin{aligned} \int_0^\infty (\coth v - 1) v^{\sigma-1} dv &= \int_0^\infty \left(\frac{e^v + e^{-v}}{e^v - e^{-v}} - 1\right) v^{\sigma-1} dv \\ &= \int_0^\infty \frac{2e^{-2v} v^{\sigma-1}}{1 - e^{-2v}} dv \end{aligned}$$

$$
\begin{aligned}
&= 2\int_0^\infty \sum_{k=1}^\infty e^{-2kv} v^{\sigma-1} dv \\
&= 2\sum_{k=1}^\infty \int_0^\infty e^{-2kv} v^{\sigma-1} dv \\
&= 2\sum_{k=1}^\infty \frac{1}{(2k)^\sigma} \Gamma(\sigma) \\
&= \frac{\Gamma(\sigma)}{2^{\sigma-1}} \zeta(\sigma), \qquad (10)
\end{aligned}
$$

where

$$\zeta(\sigma) = \sum_{k=1}^\infty \frac{1}{k^\sigma},\ \sigma > 1.$$

Lemma 3 *For* $\sigma, \widetilde{\sigma} > 1$, *we have*

$$\omega(\sigma, y) = K_2(\sigma) := \frac{\Gamma^m(\frac{1}{\alpha})}{\alpha^{m-1}\Gamma(\frac{m}{\alpha})} \frac{\Gamma(\sigma)}{2^{\sigma-1}} \zeta(\sigma)(y \in \mathbf{R}_+^n), \qquad (11)$$

$$\varpi(\sigma, x) = K_1(\sigma) := \frac{\Gamma^n(\frac{1}{\beta})}{\alpha^{n-1}\Gamma(\frac{n}{\beta})} \frac{\Gamma(\sigma)}{2^{\sigma-1}} \zeta(\sigma)(x \in \mathbf{R}_+^m), \qquad (12)$$

$$
\begin{aligned}
w(\widetilde{\sigma}, y) &:= ||y||_\beta^{-\widetilde{\sigma}} \int_{\{x \in \mathbf{R}_+^m : ||x||_\alpha \geq 1\}} \left(\coth \frac{||x||_\alpha}{||y||_\beta} - 1 \right) \frac{dx}{||x||_\alpha^{m-\widetilde{\sigma}}} \\
&= K_2(\widetilde{\sigma}) \left[1 - \theta_{\widetilde{\sigma}}(||y||_\beta) \right],
\end{aligned}
$$

and

$$
\begin{aligned}
\theta_{\widetilde{\sigma}}(||y||_\beta) &:= \frac{2^{\widetilde{\sigma}-1}}{\Gamma(\widetilde{\sigma})\zeta(\widetilde{\sigma})} \int_0^{||y||_\beta^{-1}} (\coth v - 1) v^{\widetilde{\sigma}-1} dv \\
&= O(||y||_\beta^{-\widetilde{\eta}})\ (\widetilde{\eta} > 0; y \in \mathbf{R}_+^n). \qquad (13)
\end{aligned}
$$

Proof. By (10), we obtain (11) and similarly, we get (12).
By (6) for

$$\Psi(u) = 0\ (u \in (0, 1/M^\gamma)),$$

we find

$$
\begin{aligned}
w(\widetilde{\sigma}, y) &= \frac{\Gamma^m(\frac{1}{\alpha})}{\alpha^{m-1}\Gamma(\frac{m}{\alpha})} \int_{||y||_\beta^{-1}}^\infty (\coth v - 1) v^{\widetilde{\sigma}-1} dv \\
&= \frac{\Gamma^m(\frac{1}{\alpha})}{\alpha^{m-1}\Gamma(\frac{m}{\alpha})}
\end{aligned}
$$

$$\times\left[\int_0^\infty(\coth v-1)v^{\widetilde{\sigma}-1}dv-\int_0^{||y||_\beta^{-1}}(\coth v-1)v^{\widetilde{\sigma}-1}dv\right]$$

$$=\frac{\Gamma^m(\frac{1}{\alpha})}{\alpha^{m-1}\Gamma(\frac{m}{\alpha})}\frac{\Gamma(\widetilde{\sigma})}{2^{\widetilde{\sigma}-1}}\zeta(\widetilde{\sigma})\left[1-\theta_{\widetilde{\sigma}}(||y||_\beta)\right].$$

Considering a constant $\gamma\in(1,\widetilde{\sigma})$, we obtain

$$\lim_{v\to0^+}(\coth v-1)v^\gamma=\lim_{v\to0^+}\frac{2v^\gamma}{e^{2v}-1}=\lim_{v\to0^+}\frac{2\gamma v^{\gamma-1}}{2e^{2v}}=0$$

and

$$\lim_{v\to\infty}(\coth v-1)v^\gamma=0.$$

There exists a constant $L>0$, such that

$$(\coth v-1)\le Lv^{-\gamma}.$$

Setting $\widetilde{\eta}:=\widetilde{\sigma}-\gamma\ (>0)$, it follows

$$0\le\theta_{\widetilde{\sigma}}(||y||_\beta)\le\frac{2^{\widetilde{\sigma}-1}L}{\Gamma(\widetilde{\sigma})\zeta(\widetilde{\sigma})}\int_0^{||y||_\beta^{-1}}v^{\widetilde{\eta}-1}dv=\frac{2^{\widetilde{\sigma}-1}L}{\Gamma(\widetilde{\sigma})\zeta(\widetilde{\sigma})\widetilde{\eta}}\frac{1}{||y||_\beta^{\widetilde{\eta}}},$$

and then

$$\theta_{\widetilde{\sigma}}(||y||_\beta)=O(||y||_\beta^{-\widetilde{\eta}})\ (y\in\mathbf{R}_+^n).$$

This completes the proof of the lemma.

Lemma 4 *By the assumptions of Definition 1, if* $p\in\mathbf{R}\backslash\{0,1\},\frac{1}{p}+\frac{1}{q}=1,$

$$f(x)=f(x_1,\cdots,x_m)\ge0,\ g(y)=g(y_1,\cdots,y_n)\ge0,$$

then

(i) for $p>1$, *we have the following inequality:*

$$J_1:=\left\{\int_{\mathbf{R}_+^n}\frac{||y||_\beta^{-p\sigma-n}}{[\omega(\sigma,y)]^{p-1}}\left[\int_{\mathbf{R}_+^m}\left(\coth\frac{||x||_\alpha}{||y||_\beta}-1\right)f(x)dx\right]^p dy\right\}^{\frac{1}{p}}$$

$$\le\left\{\int_{\mathbf{R}_+^m}\varpi(\sigma,x)||x||_\alpha^{p(m-\sigma)-m}f^p(x)dx\right\}^{\frac{1}{p}},\tag{14}$$

(ii) for $0<p<1$ *or* $p<0$, *we obtain the reverses of (14).*

Proof. (i) For $p>1$, by Hölder's inequality with weight (cf. [30]), it follows

$$\textstyle\int_{\mathbf{R}_+^m}\left(\coth\frac{||x||_\alpha}{||y||_\beta}-1\right)f(x)dx$$

$$= \int_{\mathbf{R}_+^m} \left(\coth \frac{||x||_\alpha}{||y||_\beta} - 1\right) \left[\frac{||x||_\alpha^{(m-\sigma)/q}}{||y||_\beta^{(n+\sigma)/p}} f(x)\right] \left[\frac{||y||_\beta^{(n+\sigma)/p}}{||x||_\alpha^{(m-\sigma)/q}}\right] dx$$

$$\leq \left\{\int_{\mathbf{R}_+^m} \left(\coth \frac{||x||_\alpha}{||y||_\beta} - 1\right) \frac{||x||_\alpha^{(m-\sigma)(p-1)}}{||y||_\beta^{n+\sigma}} f^p(x) dx\right\}^{\frac{1}{p}}$$

$$\times \left\{\int_{\mathbf{R}_+^m} \left(\coth \frac{||x||_\alpha}{||y||_\beta} - 1\right) \frac{||y||_\beta^{(n+\sigma)(q-1)}}{||x||_\alpha^{m-\sigma}} dx\right\}^{\frac{1}{q}}$$

$$= [\omega(\sigma,y)]^{\frac{1}{q}} ||y||_\beta^{\frac{n}{p}+\sigma}$$

$$\times \left\{\int_{\mathbf{R}_+^m} \left(\coth \frac{||x||_\alpha}{||y||_\beta} - 1\right) \frac{||x||_\alpha^{(m-\sigma)(p-1)}}{||y||_\beta^{n+\sigma}} f^p(x) dx\right\}^{\frac{1}{p}}. \quad (15)$$

Then by Fubini's theorem (cf. [29]), we have

$$J_1 \leq \left\{\int_{\mathbf{R}_+^n} \left[\int_{\mathbf{R}_+^m} \left(\coth \frac{||x||_\alpha}{||y||_\beta} - 1\right) \frac{||x||_\alpha^{(m-\sigma)(p-1)}}{||y||_\beta^{n+\sigma}} f^p(x) dx\right] dy\right\}^{\frac{1}{p}}$$

$$= \left\{\int_{\mathbf{R}_+^m} \left[\int_{\mathbf{R}_+^n} \left(\coth \frac{||x||_\alpha}{||y||_\beta} - 1\right) \frac{||x||_\alpha^{(m-\sigma)(p-1)}}{||y||_\beta^{n+\sigma}} dy\right] f^p(x) dx\right\}^{\frac{1}{p}}$$

$$= \left\{\int_{\mathbf{R}_+^m} \varpi(\sigma,x) ||x||_\alpha^{p(m-\sigma)-m} f^p(x) dx\right\}^{\frac{1}{p}}. \quad (16)$$

Hence, (14) follows.

(ii) For $0 < p < 1$ or $p < 0$, by the reverse Hölder inequality with weight (cf. [30]), we obtain the reverse of (15). Then by Fubini's theorem, we can still obtain the reverse of (14) and thus the lemma is proved.

Lemma 5 *By the assumptions of Lemma 4,*

(i) for $p > 1$, we have the following inequality equivalent to (14):

$$I := \int_{\mathbf{R}_+^n} \int_{\mathbf{R}_+^m} \left(\coth \frac{||x||_\alpha}{||y||_\beta} - 1\right) f(x) g(y) dx dy$$

$$\leq \left\{\int_{\mathbf{R}_+^m} \varpi(\sigma,x) ||x||_\alpha^{p(m-\sigma)-m} f^p(x) dx\right\}^{\frac{1}{p}}$$

$$\times \left\{\int_{\mathbf{R}_+^n} \omega(\sigma,y) ||y||_\beta^{q(n+\sigma)-n} g^q(y) dy\right\}^{\frac{1}{q}}, \quad (17)$$

(ii) for $0 < p < 1$ or $p < 0$, we have the reverse of (17) equivalent to the reverses of (14).

Proof. (i) For $p > 1$, by Hölder's inequality (cf. [30]), it follows that

$$
\begin{aligned}
I &= \int_{\mathbf{R}_+^n} \frac{||y||_\beta^{\frac{n}{q}-(n+\sigma)}}{[\omega(\sigma,y)]^{\frac{1}{q}}} \left[\int_{\mathbf{R}_+^m} \left(\coth \frac{||x||_\alpha}{||y||_\beta} - 1 \right) f(x)dx \right] \\
&\quad \times \left[[\omega(\sigma,y)]^{\frac{1}{q}} ||y||_\beta^{(n+\sigma)-\frac{n}{q}} g(y) \right] dy \\
&\leq J_1 \left\{ \int_{\mathbf{R}_+^n} \omega(\sigma,y) ||y||_\beta^{q(n+\sigma)-n} g^q(y) dy \right\}^{\frac{1}{q}}. \qquad (18)
\end{aligned}
$$

Then by (14), we obtain (17).

On the other hand, assuming that (17) is valid, we set

$$
g(y) := \frac{||y||_\beta^{-p\sigma-n}}{[\omega(\sigma,y)]^{p-1}} \left(\int_{\mathbf{R}_+^m} \left(\coth \frac{||x||_\alpha}{||y||_\beta} - 1 \right) f(x)dx \right)^{p-1}, y \in \mathbf{R}_+^n.
$$

Then it follows that

$$
J_1^p = \int_{\mathbf{R}_+^n} \omega(\sigma,y) ||y||_\beta^{q(n+\sigma)-n} g^q(y) dy.
$$

If $J_1 = 0$, then (14) is trivially valid; if $J_1 = \infty$, then by (16), relation (14) reduces to the form of an equality$(= \infty)$. Suppose that $0 < J_1 < \infty$. By (17), we have

$$
\begin{aligned}
0 &< \int_{\mathbf{R}_+^n} \omega(\sigma,y) ||y||_\beta^{q(n+\sigma)-n} g^q(y) dy = J_1^p = I \\
&\leq \left\{ \int_{\mathbf{R}_+^m} \varpi(\sigma,x) ||x||_\alpha^{p(m-\sigma)-m} f^p(x) dx \right\}^{\frac{1}{p}} \\
&\quad \times \left\{ \int_{\mathbf{R}_+^n} \omega(\sigma,y) ||y||_\beta^{q(n+\sigma)-n} g^q(y) dy \right\}^{\frac{1}{q}} < \infty.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
J_1 &= \left\{ \int_{\mathbf{R}_+^n} \omega(\sigma,y) ||y||_\beta^{q(n+\sigma)-n} g^q(y) dy \right\}^{\frac{1}{p}} \\
&\leq \left\{ \int_{\mathbf{R}_+^m} \varpi(\sigma,x) ||x||_\alpha^{p(m-\sigma)-m} f^p(x) dx \right\}^{\frac{1}{p}},
\end{aligned}
$$

and then (14) follows. Hence, (14) and (17) are equivalent.

(ii) For $0 < p < 1$ or $p < 0$, similarly, we obtain the reverse of (17) which is equivalent to the reverse of (14) and thus the lemma is proved.

3 Main Results and Operator Expressions

Let

$$\Phi(x) := ||x||_\alpha^{p(m-\sigma)-m},\ \Psi(y) := ||y||_\beta^{q(n+\sigma)-n}(x \in \mathbf{R}_+^m, y \in \mathbf{R}_+^n),$$

by Lemma 3, Lemma 4 and Lemma 5, we obtain

Theorem 1 *Suppose that* $\alpha, \beta > 0,\ \sigma > 1, p \in \mathbf{R}\backslash\{0,1\}, \frac{1}{p}+\frac{1}{q}=1,$

$$f(x) = f(x_1,\cdots,x_m) \geq 0,\ g(y) = g(y_1,\cdots,y_n) \geq 0,$$

$$0 < ||f||_{p,\Phi} = \left\{\int_{\mathbf{R}_+^m} \Phi(x) f^p(x)dx\right\}^{\frac{1}{p}} < \infty,$$

and

$$0 < ||g||_{q,\Psi} = \left\{\int_{\mathbf{R}_+^n} \Psi(y) g^q(y)dy\right\}^{\frac{1}{q}} < \infty.$$

(i) If $p > 1$, *then we have the following equivalent inequalities with the best possible constant factor* $K(\sigma)$, *that is*

$$I = \int_{\mathbf{R}_+^n}\int_{\mathbf{R}_+^m}\left(\coth\frac{||x||_\alpha}{||y||_\beta} - 1\right) f(x)g(y)dxdy < K(\sigma)||f||_{p,\Phi}||g||_{q,\Psi}, \tag{19}$$

and

$$J := \left\{\int_{\mathbf{R}_+^n} ||y||_\beta^{-p\sigma-n}\left(\int_{\mathbf{R}_+^m}\left(\coth\frac{||x||_\alpha}{||y||_\beta} - 1\right) f(x)dx\right)^p dy\right\}^{\frac{1}{p}} < K(\sigma)||f||_{p,\Phi}, \tag{20}$$

where

$$K(\sigma) = \left[\frac{\Gamma^n(\frac{1}{\beta})}{\beta^{n-1}\Gamma(\frac{n}{\beta})}\right]^{\frac{1}{p}}\left[\frac{\Gamma^m(\frac{1}{\alpha})}{\alpha^{m-1}\Gamma(\frac{m}{\alpha})}\right]^{\frac{1}{q}}\frac{\Gamma(\sigma)}{2^{\sigma-1}}\zeta(\sigma). \tag{21}$$

(ii) If $0 < p < 1$ *or* $p < 0$, *then we still have the equivalent reverses of (19) and (20) with the same best constant factor* $K(\sigma)$.

Proof. (i) For $p > 1$, by the given conditions, we can prove that (15) becomes a strict inequality. Otherwise if (15) takes the form of equality, then there exist constants A and B, which are not all zero, such that for a.e. $y \in \mathbf{R}_+^n$, it holds:

$$A\,\frac{||x||_\alpha^{(m-\sigma)(p-1)}}{||y||_\beta^{n+\sigma}} f^p(x) = B\,\frac{||y||_\beta^{(n+\sigma)(q-1)}}{||x||_\alpha^{m-\sigma}}\ \text{a.e. in}\ x \in \mathbf{R}_+^m. \tag{22}$$

If $A = 0$, then it follows that $B = 0$, which is impossible.
If $A \neq 0$, then (22) reduces to

$$||x||_{\alpha}^{p\,(m-\sigma)-m} f^{p}(x) = \frac{B||y||_{\beta}^{q(n+\sigma)}}{A||x||_{\alpha}^{m}} \text{ a.e. in } x \in \mathbf{R}_{+}^{m},$$

which contradicts $0 < ||f||_{p,\Phi} < \infty$.
In fact by (7), it follows

$$\int_{\mathbf{R}_{+}^{m}} ||x||_{\alpha}^{-m} dx = \infty.$$

Hence (14) still assumes the form of strict inequality. By Lemma 3 and Lemma 4, we deduce (20).

Similarly to (18), we still have

$$I \leq J \left\{ \int_{\mathbf{R}_{+}^{n}} ||y||_{\beta}^{q(n+\sigma)-n} g^{q}(y) dy \right\}^{\frac{1}{q}}. \tag{23}$$

Then by (23) and (20), we obtain (19). It is evident by Lemma 5 and the assumptions, that the relations (19) and (18) are also equivalent.

For $0 < \varepsilon < p(\sigma - 1)$, we define $\widetilde{f}(x), \widetilde{g}(y)$ as follows

$$\widetilde{f}(x) := \begin{cases} 0, & 0 < ||x||_{\alpha} < 1, \\ ||x||_{\alpha}^{\sigma - \frac{\varepsilon}{p} - m}, & ||x||_{\alpha} \geq 1, \end{cases}$$

$$\widetilde{g}(y) := \begin{cases} 0, & 0 < ||y||_{\beta} < 1, \\ ||y||_{\beta}^{-\sigma - \frac{\varepsilon}{q} - n}, & ||y||_{\beta} \geq 1. \end{cases}$$

Then for $\widetilde{\sigma} = \sigma - \frac{\varepsilon}{p}$, by (7), we derive

$$\begin{aligned} 0 &\leq \int_{\{y \in \mathbf{R}_{+}^{n} : ||y||_{\beta} \geq 1\}} ||y||_{\beta}^{-n-\varepsilon} O(||y||_{\beta}^{-\widetilde{\eta}}) dy \\ &\leq L^{*} \int_{\{y \in \mathbf{R}_{+}^{n} : ||y||_{\beta} \geq 1\}} ||y||_{\beta}^{-n-(\varepsilon+\widetilde{\eta})} dy \\ &= L^{*} \frac{\Gamma^{n}(\frac{1}{\beta})}{(\varepsilon + \widetilde{\eta})\beta^{n-1} \Gamma(\frac{n}{\beta})} < \infty, \end{aligned}$$

and in view of (7) and (13), it follows that

$$||\widetilde{f}||_{p,\Phi} ||\widetilde{g}||_{q,\Psi}$$

$$= \left\{ \int_{\{x \in \mathbf{R}_{+}^{m} : ||x||_{\alpha} \geq 1\}} ||x||_{\alpha}^{-m-\varepsilon} dx \right\}^{\frac{1}{p}} \left\{ \int_{\{y \in \mathbf{R}_{+}^{n} : ||y||_{\beta} \geq 1\}} ||y||_{\beta}^{-n-\varepsilon} dy \right\}^{\frac{1}{q}}$$

$$= \frac{1}{\varepsilon}\left\{\frac{\Gamma^m(\frac{1}{\alpha})}{\alpha^{m-1}\Gamma(\frac{m}{\alpha})}\right\}^{\frac{1}{p}}\left\{\frac{\Gamma^n(\frac{1}{\beta})}{\beta^{n-1}\Gamma(\frac{n}{\beta})}\right\}^{\frac{1}{q}},$$

and

$$\begin{aligned}
\widetilde{I} \; : \; &= \int_{\mathbf{R}_+^n}\int_{\mathbf{R}_+^m}\left(\coth\frac{||x||_\alpha}{||y||_\beta}-1\right)\widetilde{f}(x)\widetilde{g}(y)dxdy \\
&= \int_{\{y\in\mathbf{R}_+^n:||y||_\beta\geq 1\}}||y||_\beta^{-n-\varepsilon}w(\widetilde{\sigma},y)dy \\
&= K_2(\widetilde{\sigma})\int_{\{y\in\mathbf{R}_+^n:||y||_\beta\geq 1\}}||y||_\beta^{-n-\varepsilon}\left(1-O(||y||_\beta^{-\widetilde{\eta}})\right)dy \\
&= \frac{1}{\varepsilon}K_2(\widetilde{\sigma})\left[\frac{\Gamma^n(\frac{1}{\beta})}{\beta^{n-1}\Gamma(\frac{n}{\beta})}-\varepsilon O_{\widetilde{\sigma}}(1)\right].
\end{aligned}$$

If there exists a constant $K \leq K(\sigma)$, such that (19) is valid when replacing $K(\sigma)$ by K, then we obtain

$$\begin{aligned}
&\frac{\Gamma^m(\frac{1}{\alpha})}{\alpha^{m-1}\Gamma(\frac{m}{\alpha})}\frac{\Gamma(\widetilde{\sigma})}{2^{\widetilde{\sigma}-1}}\zeta(\widetilde{\sigma})\left[\frac{\Gamma^n(\frac{1}{\beta})}{\beta^{n-1}\Gamma(\frac{n}{\beta})}-\varepsilon O_{\widetilde{\sigma}}(1)\right] \\
&\leq \varepsilon\widetilde{I} < \varepsilon K||\widetilde{f}||_{p,\Phi}||\widetilde{a}||_{q,\Psi} \\
&= K\left\{\frac{\Gamma^m(\frac{1}{\alpha})}{\alpha^{m-1}\Gamma(\frac{m}{\alpha})}\right\}^{\frac{1}{p}}\left\{\frac{\Gamma^n(\frac{1}{\beta})}{\beta^{n-1}\Gamma(\frac{n}{\beta})}\right\}^{\frac{1}{q}},
\end{aligned}$$

and thus $K(\sigma) \leq K(\varepsilon \to 0^+)$.
Hence $K = K(\sigma)$ is the best possible constant factor of (19).

By the equivalency, we can prove that the constant factor $K(\sigma)$ in (20) is the best possible. Otherwise, by (23) we would reach a contradiction to the fact that the constant factor $K(\sigma)$ in (19) is the best possible.

(ii) For $0 < p < 1$ or $p < 0$, similarly, we can still obtain the equivalent reverses of (19) and (20) with the best constant factor. This completes the proof of the theorem.

Corollary 1 *Let the assumptions of Theorem 1 be fulfilled, and additionally,*

$$0 < ||f||_1 := \int_{\mathbf{R}_+^m} f(x)dx < \infty \text{ and } 0 < ||g||_1 := \int_{\mathbf{R}_+^n} g(y)dy < \infty.$$

Then,

(i) if $p > 1$, then we have the following equivalent inequalities with the best possible constant factor $K(\sigma)$, that is

$$\int_{\mathbf{R}_+^n}\int_{\mathbf{R}_+^m}\coth\frac{||x||_\alpha}{||y||_\beta}f(x)g(y)dxdy < ||f||_1||g||_1 + K(\sigma)||f||_{p,\Phi}||g||_{q,\Psi}, \qquad (24)$$

$$\left\{\int_{\mathbf{R}_+^n} ||y||_\beta^{-p\sigma-n}\left(\int_{\mathbf{R}_+^m} \coth\frac{||x||_\alpha}{||y||_\beta} f(x)dx - ||f||_1\right)^p dy\right\}^{\frac{1}{p}} < K(\sigma)||f||_{p,\Phi}; \quad (25)$$

(ii) if $0 < p < 1$ or $p < 0$, then we still have the equivalent reverses of (24) and (25) with the same best constant factor $K(\sigma)$.

For $m = n = \alpha = \beta = 1$ in Theorem 1 and Corollary 1, we obtain

Corollary 2 *Suppose that $\sigma > 1, p \in \mathbf{R}\backslash\{0,1\}, \frac{1}{p}+\frac{1}{q} = 1,$*

$$\varphi(x) := x^{p(1-\sigma)-1},\ \psi(y) := y^{q(1+\sigma)-1}\ (x,y > 0),\ f(x) \ge 0,\ g(y) \ge 0,$$

as well as

$$0 < ||f||_{p,\varphi} < \infty \text{ and } 0 < ||g||_{q,\psi} < \infty.$$

Then,

(i) for $p > 1$, we have (5) and the following equivalent inequality with the best possible constant factor

$$\frac{\Gamma(\sigma)}{2^{\sigma-1}}\zeta(\sigma),$$

that is

$$\left\{\int_0^\infty y^{-p\sigma-1}\left[\int_0^\infty \left(\coth\frac{x}{y} - 1\right) f(x)dx\right]^p dy\right\}^{\frac{1}{p}} < \frac{\Gamma(\sigma)}{2^{\sigma-1}}\zeta(\sigma)||f||_{p,\varphi}; \quad (26)$$

(ii) for $0 < p < 1$ or $p < 0$, we obtain the equivalent reverses of (5) and (26) with the same best constant factor.

Moreover, if

$$0 < ||f|| := \int_0^\infty f(x)dx < \infty \text{ and } 0 < ||g|| := \int_0^\infty g(y)dy < \infty,$$

then

(i) for $p > 1$, we have the following equivalent inequalities with the best possible constant factor

$$\frac{\Gamma(\sigma)}{2^{\sigma-1}}\zeta(\sigma),$$

that is

$$\int_0^\infty\int_0^\infty \coth\frac{x}{y} f(x)g(y)dxdy < ||f||\,||g|| + \frac{\Gamma(\sigma)}{2^{\sigma-1}}\zeta(\sigma)||f||_{p,\varphi}||g||_{q,\psi}, \quad (27)$$

$$\left\{\int_0^\infty y^{-p\sigma-1}\left[\int_0^\infty \coth\frac{x}{y} f(x)dx - ||f||\right]^p dy\right\}^{\frac{1}{p}} < \frac{\Gamma(\sigma)}{2^{\sigma-1}}\zeta(\sigma)||f||_{p,\varphi}, \quad (28)$$

(ii) for $0 < p < 1$ or $p < 0$, we obtain the equivalent reverses of (27) and (28) with the same best constant factor.

By the assumptions of Theorem 1 for $p > 1$, in view of $J < K(\sigma)||f||_{p,\Phi}$, we define:

Definition 2 *A multidimensional Hilbert-type integral operator*

$$T : L_{p,\Phi}(\mathbf{R}_+^m) \to L_{p,\Psi^{1-p}}(\mathbf{R}_+^n) \tag{29}$$

is defined as follows:

For $f \in L_{p,\Phi}(\mathbf{R}_+^m)$, *there exists a unique representation*

$$Tf \in L_{p,\Psi^{1-p}}(\mathbf{R}_+^n),$$

satisfying

$$(Tf)(y) := \int_{\mathbf{R}_+^m} \left(\coth \frac{||x||_\alpha}{||y||_\beta} - 1 \right) f(x)dx \ \ (y \in \mathbf{R}_+^n). \tag{30}$$

For $g \in L_{q,\Psi}(\mathbf{R}_+^n)$, *we define the following formal inner product of* Tf *and* g *as follows:*

$$(Tf, g) := \int_{\mathbf{R}_+^n} \int_{\mathbf{R}_+^m} \left(\coth \frac{||x||_\alpha}{||y||_\beta} - 1 \right) f(x)g(y)dxdy. \tag{31}$$

Then by Theorem 1 for

$$p > 1, \ 0 < ||f||_{p,\,\Phi}, \ ||g||_{q,\Psi} < \infty,$$

we have the following equivalent inequalities:

$$(Tf, g) < K(\sigma)||f||_{p,\,\Phi}||g||_{q,\,\Psi}, \tag{32}$$

and

$$||Tf||_{p,\,\Psi^{1-p}} < K(\sigma)||f||_{p,\,\Phi}. \tag{33}$$

It follows that the operator T is bounded with

$$||T|| := \sup_{f(\neq\theta)\in L_{p,\,\Phi}(\mathbf{R}_+^m)} \frac{||Tf||_{p,\,\Psi^{1-p}}}{||f||_{p,\,\Phi}} \leq K(\sigma).$$

Since the constant factor $K(\sigma)$ in (33) is the best possible, we obtain

$$||T|| = K(\sigma) = \left[\frac{\Gamma^n(\frac{1}{\beta})}{\beta^{n-1}\Gamma(\frac{n}{\beta})} \right]^{\frac{1}{p}} \left[\frac{\Gamma^m(\frac{1}{\alpha})}{\alpha^{m-1}\Gamma(\frac{m}{\alpha})} \right]^{\frac{1}{q}} \frac{\Gamma(\sigma)}{2^{\sigma-1}} \zeta(\sigma). \tag{34}$$

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