

# A Multidimensional Half-Discrete Hilbert-Type Inequality and the Riemann Zeta Function

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## Abstract

In this paper, by applying methods of weight functions and techniques of real analysis, a more accurate multidimensional half-discrete Hilbert's inequality with the best possible constant factor related to the Riemann zeta function is proved. Equivalent forms and some reverses are also obtained. Additionally, we consider the operator expressions with the norms and finally present a corollary related to the non-homogeneous kernel.

**Key words:** Gamma function; Riemann zeta function; numerical estimates; half-discrete Hilbert's inequality; weight function; Hermite-Hadamard's inequality; Hilbert's operator; equivalent form;

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## 1 Introduction

Assuming that  $p > 1, \frac{1}{p} + \frac{1}{q} = 1, f(x), g(y) \geq 0, f \in L^p(\mathbf{R}_+), g \in L^q(\mathbf{R}_+)$ ,

$$\|f\|_p = \left( \int_0^\infty f^p(x) dx \right)^{\frac{1}{p}} > 0, \|g\|_q > 0,$$

we have the following Hardy-Hilbert's integral inequality (cf. [1]):

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} dx dy < \frac{\pi}{\sin(\pi/p)} \|f\|_p \|g\|_q, \quad (1.1)$$

where the constant factor  $\pi / \sin(\pi/p)$  is the best possible.

If  $a_m, b_n \geq 0, a = \{a_m\}_{m=1}^\infty \in l^p, b = \{b_n\}_{n=1}^\infty \in l^q, \|a\|_p = (\sum_{m=1}^\infty a_m^p)^{\frac{1}{p}} > 0, \|b\|_q > 0$ , then we still have the following discrete variant of the above inequality with the same best constant factor  $\pi / \sin(\pi/p)$ :

$$\sum_{m=1}^\infty \sum_{n=1}^\infty \frac{a_m b_n}{m+n} < \frac{\pi}{\sin(\pi/p)} \|a\|_p \|b\|_q \quad (1.2)$$

Inequalities (1.1) and (1.2) are important in Analysis and its applications (cf. [1], [2], [3], [4], [5], [31]).

In 1998, by introducing an independent parameter  $\lambda \in (0, 1]$ , Yang [6] presented an extension of (1.1) for  $p = q = 2$ . In 2009-2011, Yang [3], [4] obtained some extensions of (1.1) and (1.2) as follows:

If  $\lambda_1, \lambda_2, \lambda \in \mathbf{R}, \lambda_1 + \lambda_2 = \lambda, k_\lambda(x, y)$  is a non-negative homogeneous function of degree  $-\lambda$ ,

$$k(\lambda_1) = \int_0^\infty k_\lambda(t, 1) t^{\lambda_1 - 1} dt \in \mathbf{R}_+, \phi(x) = x^{p(1-\lambda_1)-1}, \psi(y) = y^{q(1-\lambda_2)-1}, f(x), g(y) \geq 0,$$

$$f \in L_{p,\phi}(\mathbf{R}_+) = \left\{ f; \|f\|_{p,\phi} := \left( \int_0^\infty \phi(x)|f(x)|^p dx \right)^{\frac{1}{p}} < \infty \right\},$$

$$g \in L_{q,\psi}(\mathbf{R}_+), \|f\|_{p,\phi}, \|g\|_{q,\psi} > 0,$$

then we have

$$\int_0^\infty \int_0^\infty k_\lambda(x, y) f(x) g(y) dx dy < k(\lambda_1) \|f\|_{p,\phi} \|g\|_{q,\psi}, \quad (1.3)$$

where the constant factor  $k(\lambda_1)$  is the best possible. Moreover, if  $k_\lambda(x, y)$  is finite and  $k_\lambda(x, y)x^{\lambda_1-1}$  ( $k_\lambda(x, y)y^{\lambda_2-1}$ ) is decreasing with respect to  $x > 0$  ( $y > 0$ ), then for  $a_m, b_n \geq 0$ ,

$$a \in l_{p,\phi} = \left\{ a; \|a\|_{p,\phi} := \left( \sum_{n=1}^\infty \phi(n)|a_n|^p \right)^{\frac{1}{p}} < \infty \right\},$$

$$b = \{b_n\}_{n=1}^\infty \in l_{q,\psi}, \|a\|_{p,\phi}, \|b\|_{q,\psi} > 0,$$

we have

$$\sum_{m=1}^\infty \sum_{n=1}^\infty k_\lambda(m, n) a_m b_n < k(\lambda_1) \|a\|_{p,\phi} \|b\|_{q,\psi}, \quad (1.4)$$

where, the constant factor  $k(\lambda_1)$  is still the best possible.

Clearly, for

$$\lambda = 1, k_1(x, y) = \frac{1}{x+y}, \lambda_1 = \frac{1}{q}, \lambda_2 = \frac{1}{p},$$

(1.3) reduces to (1.1), while (1.4) reduces to (1.2). Some other results including the multi-dimensional Hilbert-type integral inequalities are provided in [7]–[20].

Regarding the topic of half-discrete Hilbert-type inequalities with non-homogeneous kernels, Hardy, Littlewood and Pólya provided a few results in Theorem 351 of [1]. However, it was not proved that the constant factors are the best possible. Yang [21] presented a result with the kernel  $1/(1+nx)^\lambda$  by introducing an interval variable and proved that the constant factor is the best possible. In 2011 Yang [22] proved the following half-discrete Hardy-Hilbert's inequality with the best possible constant factor  $B(\lambda_1, \lambda_2)$ :

$$\int_0^\infty f(x) \sum_{n=1}^\infty \frac{a_n}{(x+n)^\lambda} dx < B(\lambda_1, \lambda_2) \|f\|_{p,\phi} \|a\|_{q,\psi}, \quad (1.5)$$

where,

$$\lambda_1 \lambda_2 > 0, 0 \leq \lambda_2 \leq 1, \lambda_1 + \lambda_2 = \lambda, B(u, v) = \int_0^\infty \frac{1}{(1+t)^{u+v}} t^{u-1} dt \quad (u, v > 0)$$

is the beta function. Zhong et al ([23]–[28]) investigated several half-discrete Hilbert-type inequalities with particular kernels.

Using the method of weight functions and techniques of discrete and integral Hilbert-type inequalities with some additional conditions on the kernel, the following half-discrete Hilbert-type inequality with a general homogeneous kernel of degree  $-\lambda \in \mathbf{R}$  and the best constant factor  $k(\lambda_1)$  is obtained

$$\int_0^\infty f(x) \sum_{n=1}^\infty k_\lambda(x, n) a_n dx < k(\lambda_1) \|f\|_{p,\phi} \|a\|_{q,\psi}, \quad (1.6)$$

which is an extension of (1.5) (see Yang and Chen [29]). Moreover, a half-discrete Hilbert-type inequality with a general non-homogeneous kernel and the best constant factor is given by Yang [30].

**Remark.** (1) Several different kinds of Hilbert-type discrete, half-discrete and integral inequalities with applications have been proved during the last twenty years. In this paper, special attention is given to new results proved during 2009–2012. Several generalizations, extensions and refinements of Hilbert-type discrete, half-discrete and integral inequalities involving many special functions such as beta, gamma, hypergeometric, trigonometric, hyperbolic, Riemann zeta, Bernoulli functions, Bernoulli numbers, Euler constant etc., are discussed.

(2) Yang presented several new results on Hilbert-type operators with general homogeneous kernels of degree real numbers and two pairs of conjugate exponents as well as the related inequalities (see [3], [4], [5], [31], [32]).

In this paper, by the use of the method of weight functions and techniques of real analysis, a more accurate multidimensional half-discrete Hilbert-type inequality with the best possible constant factor related to the Riemann zeta function is provided. Equivalent forms and some reverses are obtained. We also consider the operator expressions with the norm and a corollary related to the non-homogeneous kernel.

## 2 Some Lemmas

**Lemma 2.1.** Suppose that  $(-1)^i h^{(i)}(t) > 0 (t > 0; i = 0, 1, 2)$ . Then

(i) for  $b > 0, 0 < \alpha \leq 1$ , we have

$$(-1)^i \frac{d^i}{dx^i} h((b+x^\alpha)^{\frac{1}{\alpha}}) > 0 \quad (x > 0; i = 1, 2);$$

(ii) for  $\int_{\frac{1}{2}}^\infty h(t) dt < \infty$ , we have

$$\int_1^\infty h(t) dt < \sum_{n=1}^\infty h(n) < \int_{\frac{1}{2}}^\infty h(t) dt. \quad (2.1)$$

*Proof.* (i) It holds

$$\begin{aligned} \frac{d}{dx} h((b+x^\alpha)^{\frac{1}{\alpha}}) &= h'((b+x^\alpha)^{\frac{1}{\alpha}})(b+x^\alpha)^{\frac{1}{\alpha}-1} x^{\alpha-1} < 0, \\ \frac{d^2}{dx^2} h((b+x^\alpha)^{\frac{1}{\alpha}}) &= \frac{d}{dx} [h'((b+x^\alpha)^{\frac{1}{\alpha}})(b+x^\alpha)^{\frac{1}{\alpha}-1} x^{\alpha-1}] \end{aligned}$$

$$\begin{aligned}
&= h''((b+x^\alpha)^{\frac{1}{\alpha}})(b+x^\alpha)^{\frac{2}{\alpha}-2}x^{2\alpha-2} \\
&\quad + \alpha(\frac{1}{\alpha}-1)h'((b+x^\alpha)^{\frac{1}{\alpha}})(b+x^\alpha)^{\frac{1}{\alpha}-2}x^{2\alpha-2} \\
&\quad + (\alpha-1)h'((b+x^\alpha)^{\frac{1}{\alpha}})(b+x^\alpha)^{\frac{1}{\alpha}-1}x^{\alpha-2} \\
&= h''((b+x^\alpha)^{\frac{1}{\alpha}})(b+x^\alpha)^{\frac{2}{\alpha}-2}x^{2\alpha-2} \\
&\quad + b(\alpha-1)h'((b+x^\alpha)^{\frac{1}{\alpha}})(b+x^\alpha)^{\frac{1}{\alpha}-2}x^{\alpha-2} > 0.
\end{aligned}$$

(ii) Since  $h(t)$  is a decreasing convex function, by Hermite-Hadamard's inequality (cf. [34]), we have

$$\int_n^{n+1} h(t)dt < h(n) < \int_{n-\frac{1}{2}}^{n+\frac{1}{2}} h(t)dt \quad (n \in \mathbb{N}),$$

and thus

$$\begin{aligned}
\int_1^\infty h(t)dt &= \sum_{n=1}^\infty \int_n^{n+1} h(t)dt < \sum_{n=1}^\infty h(n) \\
&< \sum_{n=1}^\infty \int_{n-\frac{1}{2}}^{n+\frac{1}{2}} h(t)dt = \int_{\frac{1}{2}}^\infty h(t)dt.
\end{aligned}$$

Hence (2.1) follows.  $\square$

**Note.** If  $(-1)^i h^{(i)}(t) > 0 (t > 0; i = 0, 1)$ ,  $\int_0^\infty h(t)dt < \infty$ , since  $h(n) < \int_{n-1}^n h(t)dt$  ( $n \in \mathbb{N}$ ), we still have

$$\sum_{n=1}^\infty h(n) < \sum_{n=1}^\infty \int_{n-1}^n h(t)dt = \int_0^\infty h(t)dt.$$

If  $i_0, j_0 \in \mathbb{N}, \alpha, \beta > 0$ , we define

$$\begin{aligned}
||x||_\alpha &:= \left( \sum_{k=1}^{i_0} |x_k|^\alpha \right)^{\frac{1}{\alpha}} \quad (x = (x_1, \dots, x_{i_0}) \in \mathbf{R}^{i_0}), \\
||y||_\beta &:= \left( \sum_{k=1}^{j_0} |y_k|^\beta \right)^{\frac{1}{\beta}} \quad (y = (y_1, \dots, y_{j_0}) \in \mathbf{R}^{j_0}). \tag{2.2}
\end{aligned}$$

**Lemma 2.2.** If  $s \in \mathbb{N}, \gamma, M > 0, \Psi(u)$  is a non-negative measurable function in  $(0, 1]$ , and  $D_M := \left\{ x \in \mathbf{R}_+^s; 0 < u = \sum_{i=1}^s \left(\frac{x_i}{M}\right)^\gamma \leq 1 \right\}$ , then we have (cf. [31])

$$\int \cdots \int_{D_M} \Psi \left( \sum_{i=1}^s \left(\frac{x_i}{M}\right)^\gamma \right) dx_1 \cdots dx_s = \frac{M^s \Gamma^s(\frac{1}{\gamma})}{\gamma^s \Gamma(\frac{s}{\gamma})} \int_0^1 \Psi(u) u^{\frac{s}{\gamma}-1} du. \tag{2.3}$$

In view of (2.3) and the condition, it follows that:

(i) For  $\mathbf{R}_+^s = \left\{ x \in \mathbf{R}_+^s; 0 < u = \sum_{i=1}^s \left(\frac{x_i}{M}\right)^\gamma \leq 1 (M \rightarrow \infty) \right\}$ , we have

$$\int \cdots \int_{\mathbf{R}_+^s} \Psi \left( \sum_{i=1}^s \left(\frac{x_i}{M}\right)^\gamma \right) dx_1 \cdots dx_s = \lim_{M \rightarrow \infty} \frac{M^s \Gamma^s(\frac{1}{\gamma})}{\gamma^s \Gamma(\frac{s}{\gamma})} \int_0^1 \Psi(u) u^{\frac{s}{\gamma}-1} du; \tag{2.4}$$

(ii) for  $\{x \in \mathbf{R}_+^s; ||x||_\gamma \geq 1\} = \left\{ x \in \mathbf{R}_+^s; M^{-\gamma} < u = \sum_{i=1}^s \left(\frac{x_i}{M}\right)^\gamma \leq 1 \right. \\ (M \rightarrow \infty) \},$  setting  $\Psi(u) = 0 (u \in (0, M^{-\gamma}))$ , we have

$$\int \cdots \int_{\{||x||_\gamma \geq 1\}} \Psi \left( \sum_{i=1}^s \left(\frac{x_i}{M}\right)^\gamma \right) dx_1 \cdots dx_s, \quad (M > 1, M \rightarrow \infty) \\ = \lim_{M \rightarrow \infty} \frac{M^s \Gamma^s(\frac{1}{\gamma})}{\gamma^s \Gamma(\frac{s}{\gamma})} \int_{M^{-\gamma}}^1 \Psi(u) u^{\frac{s}{\gamma}-1} du; \quad (2.5)$$

(iii) for  $\{x \in \mathbf{R}_+^s; ||x||_\gamma \leq 1\} = \left\{ x \in \mathbf{R}_+^s; 0 < u = \sum_{i=1}^s \left(\frac{x_i}{M}\right)^\gamma \leq \frac{1}{M^\gamma} \right\},$  setting  $\Psi(u) = 0 (u \in (\frac{1}{M^\gamma}, \infty))$ , we have

$$\int \cdots \int_{\{||x||_\gamma \leq 1\}} \Psi \left( \sum_{i=1}^s \left(\frac{x_i}{M}\right)^\gamma \right) dx_1 \cdots dx_s \\ = \frac{M^s \Gamma^s(\frac{1}{\gamma})}{\gamma^s \Gamma(\frac{s}{\gamma})} \int_0^{M^{-\gamma}} \Psi(u) u^{\frac{s}{\gamma}-1} du; \quad (2.6)$$

(iv) for

$$\{x \in \mathbf{R}_+^s; x_i \geq 1\} = \left\{ x \in \mathbf{R}_+^s; \frac{s}{M^\gamma} < u = \sum_{i=1}^s \left(\frac{x_i}{M}\right)^\gamma \leq 1 (M^\gamma > s; M \rightarrow \infty) \right\},$$

setting  $\Psi(u) = 0 (u \in (0, \frac{s}{M^\gamma}))$ , we have

$$\int \cdots \int_{\{x_i \geq 1\}} \Psi \left( \sum_{i=1}^s \left(\frac{x_i}{M}\right)^\gamma \right) dx_1 \cdots dx_s \\ = \lim_{M \rightarrow \infty} \frac{M^s \Gamma^s(\frac{1}{\gamma})}{\gamma^s \Gamma(\frac{s}{\gamma})} \int_{sM^{-\gamma}}^1 \Psi(u) u^{\frac{s}{\gamma}-1} du; \quad (2.7)$$

**Lemma 2.3.** If  $s \in \mathbb{N}, \gamma > 0, \epsilon > 0, c = (c_1, \dots, c_s) \in [0, \frac{1}{2}]^s$ , then we have

$$\int \cdots \int_{\{x \in \mathbf{R}_+^s; x_i \geq 1\}} ||x||_\gamma^{-s-\epsilon} dx_1 \cdots dx_s = \frac{\Gamma^s(\frac{1}{\gamma})}{\epsilon s^{\epsilon/\gamma} \gamma^{s-1} \Gamma(\frac{s}{\gamma})}, \quad (2.8)$$

$$\sum_m ||m - c||_\gamma^{-s-\epsilon} = \frac{\Gamma^s(\frac{1}{\gamma})}{\epsilon s^{\epsilon/\gamma} \gamma^{s-1} \Gamma(\frac{s}{\gamma})} + O(1)(\epsilon \rightarrow 0^+), \quad (2.9)$$

*Proof.* By (2.7), it follows that

$$\int \cdots \int_{\{x \in \mathbf{R}_+^s; x_i \geq 1\}} ||x||_\gamma^{-s-\epsilon} dx_1 \cdots dx_s \\ = \int \cdots \int_{\{x \in \mathbf{R}_+^s; x_i \geq 1\}} \left( M \left( \sum_{i=1}^s \left(\frac{x_i}{M}\right)^\gamma \right)^{\frac{1}{\gamma}} \right)^{-s-\epsilon} dx_1 \cdots dx_s$$

$$= \lim_{M \rightarrow \infty} \frac{M^s \Gamma^s(\frac{1}{\gamma})}{\gamma^s \Gamma(\frac{s}{\gamma})} \int_{sM^{-\gamma}}^1 (Mu^{1/\gamma})^{-s-\epsilon} u^{\frac{s}{\gamma}-1} du = \frac{\Gamma^s(\frac{1}{\gamma})}{\epsilon s^{\epsilon/\gamma} \gamma^{s-1} \Gamma(\frac{s}{\gamma})}. \quad (2.10)$$

Hence we have (2.8).

By (2.1) and (2.8), we obtain

$$\begin{aligned} \sum_m ||m - c||_\gamma^{-s-\epsilon} &> \int_{\{x \in \mathbf{R}_+^s; x_i \geq 1+c_i\}} ||x - c||_\gamma^{-s-\epsilon} dx = \int_{\{u \in \mathbf{R}_+^s; u_i \geq 1\}} ||u||_\gamma^{-s-\epsilon} du \\ &= \lim_{M \rightarrow \infty} \int \cdots \int_{D_M} \left( M \left( \sum_{i=1}^s \left( \frac{u_i}{M} \right)^\gamma \right)^{\frac{1}{\gamma}} \right)^{-s-\epsilon} du_1 \cdots du_s \\ &= \lim_{M \rightarrow \infty} \frac{M^s \Gamma^s(\frac{1}{\gamma})}{\gamma^s \Gamma(\frac{s}{\gamma})} \int_{sM^{-\gamma}}^1 (Mv^{1/\gamma})^{-s-\epsilon} v^{\frac{s}{\gamma}-1} dv = \frac{\Gamma^s(\frac{1}{\gamma})}{\epsilon s^{\epsilon/\gamma} \gamma^{s-1} \Gamma(\frac{s}{\gamma})}. \end{aligned} \quad (2.11)$$

For  $s = 1, 0 < \sum_{m=1}^{+\infty} ||m - c||_\gamma^{-1-\epsilon} < \infty$ ; for  $s \geq 2$ , by (2.7), we get

$$\begin{aligned} 0 &< \sum_{\{m \in \mathbf{N}^s; \exists i_0, m_{i_0} = 1, 2\}} ||m - c||_\gamma^{-s-\epsilon} \leq a + \sum_{\{m \in \mathbf{N}^{s-1}; m_i \geq 3\}} ||m - c||_\gamma^{-(s-1)-(1+\epsilon)} \\ &\leq a + \int_{\{x \in \mathbf{R}_+^{s-1}; x_i \geq 1+c_i\}} ||x - c||_\gamma^{-(s-1)-(1+\epsilon)} dx \\ &= a + \int_{\{u \in \mathbf{R}_+^{s-1}; u_i \geq 1\}} ||u||_\gamma^{-(s-1)-(1+\epsilon)} du \\ &= a + \frac{\Gamma^{s-1}(\frac{1}{\gamma})}{(1+\epsilon)(s-1)^{(1+\epsilon)/\gamma} \gamma^{s-2} \Gamma(\frac{s-1}{\gamma})} < +\infty \quad (a \in \mathbf{R}_+). \end{aligned}$$

Thus we obtain

$$\begin{aligned} 0 &< \sum_{\{m \in \mathbf{N}^s; m_i \geq 1\}} ||m - c||_\gamma^{-s-\epsilon} \\ &= \sum_{\{m \in \mathbf{N}^s; \exists i_0, m_{i_0} = 1, 2\}} ||m - c||_\gamma^{-s-\epsilon} + \sum_{\{m \in \mathbf{N}^s; m_i \geq 3\}} ||m - c||_\gamma^{-s-\epsilon} \\ &\leq \tilde{O}(1) + \int_{\{x \in \mathbf{R}_+^s; x_i \geq 1+c_i\}} ||x - c||_\gamma^{-s-\epsilon} dx \\ &= \tilde{O}(1) + \frac{\Gamma^s(\frac{1}{\gamma})}{\epsilon s^{\epsilon/\gamma} \gamma^{s-1} \Gamma(\frac{s}{\gamma})} \quad (\epsilon \rightarrow 0^+). \end{aligned} \quad (2.12)$$

In view of (2.11) and (2.12), (2.9) follows.  $\square$

### 3 A More Accurate Multidimensional Hilbert-Type Inequality with Parameters

In this section, we set  $i_0, j_0 \in \mathbf{N}$ ,  $\alpha, \beta > 0$ ,  $p \in \mathbf{R} \setminus \{0, 1\}$ ,  $\frac{1}{p} + \frac{1}{q} = 1$ .

**Definition 3.1.** Let  $x = (x_1, \dots, x_{i_0}) \in \mathbf{R}_+^{i_0}$ ,  $n = (n_1, \dots, n_{j_0}) \in \mathbf{N}^{j_0}$ ,  $\tau = (\tau_1, \dots, \tau_{j_0}) \in [0, \frac{1}{2}]^{j_0}$ ,  $\delta \in \{-1, 1\}$ ,  $\rho > 0$ ,  $0 < \eta < \sigma < j_0$ ,  $\csc h(u) := 2/(e^u - e^{-u})$  be the hyperbolic cosecant function. Let the two weight coefficients  $\omega_\delta(\sigma, n)$  and  $\varpi_\delta(\sigma, x)$  be defined as follows:

$$\omega_\delta(\sigma, n) : = \|n - \tau\|_\beta^\sigma \int_{\mathbf{R}_+^{i_0}} \csc h \left( \frac{\rho \|n - \tau\|_\beta^\eta}{\|x\|_\alpha^{\delta\eta}} \right) \|x\|_\alpha^{-\delta\sigma-i_0} dx, \quad (3.1)$$

$$\varpi_\delta(\sigma, x) : = \|x\|_\alpha^{-\delta\sigma} \sum_n \csc h \left( \frac{\rho \|n - \tau\|_\beta^\eta}{\|x\|_\alpha^{\delta\eta}} \right) \|n - \tau\|_\beta^{\sigma-j_0}, \quad (3.2)$$

where,  $\sum_n = \sum_{n_{j_0}=1}^\infty \cdots \sum_{n_1=1}^\infty$ .

**Lemma 3.2.** By the assumptions of Definition 1, if both  $\omega_\delta(\sigma, n)$  and  $\varpi_\delta(\sigma, x)$  are finite,  $f(x) = f(x_1, \dots, x_{i_0}) \geq 0$ ,  $a_n = a_{(n_1, \dots, n_{j_0})} \geq 0$ , then

(i) for  $p > 1$ , we have the following inequality:

$$\begin{aligned} J_1 &:= \left( \sum_n \frac{\|n - \tau\|_\beta^{p\sigma-j_0}}{[\omega_\delta(\sigma, n)]^{p-1}} \left( \int_{\mathbf{R}_+^{i_0}} \csc h \left( \frac{\rho \|n - \tau\|_\beta^\eta}{\|x\|_\alpha^{\delta\eta}} \right) f(x) dx \right)^p \right)^{\frac{1}{p}} \\ &\leq \left( \int_{\mathbf{R}_+^{i_0}} \varpi_\delta(\sigma, x) \|x\|_\alpha^{p(i_0+\delta\sigma)-i_0} f^p(x) dx \right)^{\frac{1}{p}}, \end{aligned} \quad (3.3)$$

$$\begin{aligned} J_2 &:= \left( \int_{\mathbf{R}_+^{i_0}} \frac{\|x\|_\alpha^{-\delta q \sigma - i_0}}{(\varpi_\delta(\sigma, x))^q} \left( \sum_n \csc h \left( \frac{\rho \|n - \tau\|_\beta^\eta}{\|x\|_\alpha^{\delta\eta}} \right) a_n \right)^q dx \right)^{\frac{1}{q}} \\ &\leq \left( \sum_n \omega_\delta(\sigma, n) \|n - \tau\|_\beta^{q(j_0-\sigma)-j_0} a_n^q \right)^{\frac{1}{q}}; \end{aligned} \quad (3.4)$$

(ii) for  $p < 0$ , or  $0 < p < 1$ , we have the reverses of (3.3) and (3.4).

*Proof.* (i) For  $p > 1$ , by the weighted Hölder's inequality (cf. [34]), it follows

$$\begin{aligned} &\int_{\mathbf{R}_+^{i_0}} \csc h \left( \frac{\rho \|n - \tau\|_\beta^\eta}{\|x\|_\alpha^{\delta\eta}} \right) f(x) dx \\ &= \int_{\mathbf{R}_+^{i_0}} \csc h \left( \frac{\rho \|n - \tau\|_\beta^\eta}{\|x\|_\alpha^{\delta\eta}} \right) \left( \frac{\|x\|_\alpha^{(i_0+\delta\sigma)/q} f(x)}{\|n - \tau\|_\beta^{(j_0-\sigma)/p}} \right) \left( \frac{\|n - \tau\|_\beta^{(j_0-\sigma)/p}}{\|x\|_\alpha^{(i_0+\delta\sigma)/q}} \right) dx \end{aligned}$$

$$\begin{aligned}
&\leq \left( \int_{\mathbf{R}_+^{i_0}} \csc h \left( \frac{\rho ||n-\tau||_\beta^\eta}{||x||_\alpha^{\delta\eta}} \right) \frac{||x||_\alpha^{(i_0+\delta\sigma)(p-1)}}{||n-\tau||_\beta^{j_0-\sigma}} f^p(x) dx \right)^{\frac{1}{p}} \\
&\quad \times \left( \int_{\mathbf{R}_+^{i_0}} \csc h \left( \frac{\rho ||n-\tau||_\beta^\eta}{||x||_\alpha^{\delta\eta}} \right) \frac{||n-\tau||_\beta^{(j_0-\sigma)(q-1)}}{||x||_\alpha^{i_0+\delta\sigma}} dx \right)^{\frac{1}{q}} \\
&= (\omega_\delta(\sigma, n))^{\frac{1}{q}} ||n-\tau||_\beta^{\frac{j_0}{p}-\sigma} \\
&\quad \times \left( \int_{\mathbf{R}_+^{i_0}} \csc h \left( \frac{\rho ||n-\tau||_\beta^\eta}{||x||_\alpha^{\delta\eta}} \right) \frac{||x||_\alpha^{(i_0+\delta\sigma)(p-1)}}{||n-\tau||_\beta^{j_0-\sigma}} f^p(x) dx \right)^{\frac{1}{p}}. \tag{3.5}
\end{aligned}$$

Then by the Lebesgue term by term integration theorem (cf. [35]), we have

$$\begin{aligned}
J_1 &\leq \left( \sum_n \int_{\mathbf{R}_+^{i_0}} \csc h \left( \frac{\rho ||n-\tau||_\beta^\eta}{||x||_\alpha^{\delta\eta}} \right) \frac{||x||_\alpha^{(i_0+\delta\sigma)(p-1)}}{||n-\tau||_\beta^{j_0-\sigma}} f^p(x) dx \right)^{\frac{1}{p}} \\
&= \left( \int_{\mathbf{R}_+^{i_0}} \sum_n \csc h \left( \frac{\rho ||n-\tau||_\beta^\eta}{||x||_\alpha^{\delta\eta}} \right) \frac{||x||_\alpha^{(i_0+\delta\sigma)(p-1)}}{||n-\tau||_\beta^{j_0-\sigma}} f^p(x) dx \right)^{\frac{1}{p}} \\
&= \left( \int_{\mathbf{R}_+^{i_0}} \varpi_\delta(\sigma, x) ||x||_\alpha^{p(i_0+\delta\sigma)-i_0} f^p(x) dx \right)^{\frac{1}{p}}. \tag{3.6}
\end{aligned}$$

Hence, (3.3) follows. Similarly, we obtain

$$\begin{aligned}
\sum_n \csc h \left( \frac{\rho ||n-\tau||_\beta^\eta}{||x||_\alpha^{\delta\eta}} \right) a_n &\leq (\varpi_\delta(\sigma, x))^{\frac{1}{p}} ||x||_\alpha^{\frac{i_0}{q}+\delta\sigma} \\
&\quad \times \left( \sum_n \csc h \left( \frac{\rho ||n-\tau||_\beta^\eta}{||x||_\alpha^{\delta\eta}} \right) \frac{||n-\tau||_\beta^{(j_0-\sigma)(q-1)}}{||x||_\alpha^{i_0+\delta\sigma}} a_n^q \right)^{\frac{1}{q}}.
\end{aligned}$$

Thus, by the Lebesgue term by term integration theorem, similarly to the way we obtained (3.6), we get (3.4).

(ii) For  $p < 0$ , or  $0 < p < 1$ , by the reverse weighted Hölder's inequality (cf. [34]), we obtain the reverse of (3.5). Then by the Lebesgue term by term integration theorem, we can obtain the reverse of (3.4).  $\square$

**Lemma 3.3.** *With the assumptions of Lemma 3.2, we have*

(i) *for  $p > 1$ , the following inequality equivalent to (3.3) and (3.4) holds true*

$$\begin{aligned}
I &:= \sum_n \int_{\mathbf{R}_+^{i_0}} \csc h \left( \frac{\rho ||n-\tau||_\beta^\eta}{||x||_\alpha^{\delta\eta}} \right) a_n f(x) dx \\
&\leq \left( \int_{\mathbf{R}_+^{i_0}} \varpi_\delta(\sigma, x) ||x||_\alpha^{p(i_0+\delta\sigma)-i_0} f^p(x) dx \right)^{\frac{1}{p}} \left( \sum_n \omega_\delta(\sigma, n) ||n-\tau||_\beta^{q(j_0-\sigma)-j_0} a_n^q \right)^{\frac{1}{q}} \tag{3.7}
\end{aligned}$$

(ii) for  $p < 0$ , or  $0 < p < 1$ , we obtain the reverse of (3.7) which is equivalent to the reverses of (3.3) and (3.4).

*Proof.* (i) For  $p > 1$ , by Hölder's inequality (cf. [34]), it follows

$$\begin{aligned} I &= \sum_n \frac{\|n - \tau\|_{\beta}^{\frac{j_0}{q} - (j_0 - \sigma)}}{[\omega_{\delta}(\sigma, n)]^{\frac{1}{q}}} \left( \int_{\mathbf{R}_+^{i_0}} \csc h \left( \frac{\rho \|n - \tau\|_{\beta}^{\eta}}{\|x\|_{\alpha}^{\delta\eta}} \right) f(x) dx \right) \\ &\quad \times \left( (\omega_{\delta}(\sigma, n))^{\frac{1}{q}} \|n - \tau\|_{\beta}^{(j_0 - \sigma) - \frac{j_0}{q}} a_n \right) \\ &\leq J_1 \left( \sum_n \omega_{\delta}(\sigma, n) \|n - \tau\|_{\beta}^{q(j_0 - \sigma) - j_0} a_n^q \right)^{\frac{1}{q}}. \end{aligned}$$

Then by (3.3), we obtain (3.7).

On the other hand, assuming that (3.7) is valid, we set

$$a_n := \frac{\|n - \tau\|_{\beta}^{p\sigma - j_0}}{[\omega_{\delta}(\sigma, n)]^{p-1}} \left( \int_{\mathbf{R}_+^{i_0}} \csc h \left( \frac{\rho \|n - \tau\|_{\beta}^{\eta}}{\|x\|_{\alpha}^{\delta\eta}} \right) f(x) dx \right)^{p-1}, n \in \mathbf{N}^{j_0}.$$

Then it follows that

$$J_1^p = \sum_n \omega_{\delta}(\sigma, n) \|n - \tau\|_{\beta}^{q(j_0 - \sigma) - j_0} a_n^q.$$

If  $J_1 = 0$ , then (3.3) is trivially valid;

if  $J_1 = \infty$ , then by (3.6), inequality (3.3) reduces to an equality ( $= \infty$ ).

Suppose that  $0 < J_1 < \infty$ . By (3.7), we obtain

$$\begin{aligned} 0 &< \sum_n \omega_{\delta}(\sigma, n) \|n - \tau\|_{\beta}^{q(j_0 - \sigma) - j_0} a_n^q = J_1^p = I \\ &\leq \left( \int_{\mathbf{R}_+^{i_0}} \varpi_{\delta}(\sigma, x) \|x\|_{\alpha}^{p(i_0 + \delta\sigma) - i_0} f^p(x) dx \right)^{\frac{1}{p}} \\ &\quad \times \left( \sum_n \omega_{\delta}(\sigma, n) \|n - \tau\|_{\beta}^{q(j_0 - \sigma) - j_0} a_n^q \right)^{\frac{1}{q}} < \infty. \end{aligned}$$

Therefore

$$\begin{aligned} J_1 &= \left( \sum_n \omega_{\delta}(\sigma, n) \|n - \tau\|_{\beta}^{q(j_0 - \sigma) - j_0} a_n^q \right)^{\frac{1}{p}} \\ &\leq \left( \int_{\mathbf{R}_+^{i_0}} \varpi_{\delta}(\sigma, x) \|x\|_{\alpha}^{p(i_0 + \delta\sigma) - i_0} f^p(x) dx \right)^{\frac{1}{p}}, \end{aligned}$$

and then (3.3) follows. Hence, (3.3) and (3.7) are equivalent.

Similarly, by Hölder's inequality we obtain

$$I \leq \left( \int_{\mathbf{R}_+^{i_0}} \varpi_{\delta}(\sigma, x) \|x\|_{\alpha}^{p(i_0 + \delta\sigma) - i_0} f^p(x) dx \right)^{\frac{1}{p}} J_2.$$

Then by (3.4), we have (3.7). On the other hand, assuming that (3.7) is true, we set

$$f(x) = \frac{\|x\|_\alpha^{-q\delta\sigma-i_0}}{[\varpi_\delta(\sigma, x)]^{q-1}} \left( \sum_n \csc h \left( \frac{\rho \|n - \tau\|_\beta^\eta}{\|x\|_\alpha^{\delta\eta}} \right)^\eta a_n \right)^{q-1} (x \in \mathbf{R}_+^{i_0}).$$

Then, it follows

$$J_2^q = \int_{\mathbf{R}_+^{i_0}} \varpi_\delta(\sigma, x) \|x\|_\alpha^{p(i_0 + \delta\sigma) - i_0} f^p(x) dx.$$

Similarly, by (3.7), we get

$$\begin{aligned} J_2 &= \left( \int_{\mathbf{R}_+^{i_0}} \varpi_\delta(\sigma, x) \|x\|_\alpha^{p(i_0 + \delta\sigma) - i_0} f^p(x) dx \right)^{\frac{1}{q}} \\ &\leq \left( \sum_n \omega_\delta(\sigma, n) \|n - \tau\|_\beta^{q(j_0 - \sigma) - j_0} a_n^q \right)^{\frac{1}{q}}, \end{aligned}$$

and then (3.4) is equivalent to (3.7).

Hence (3.3), (3.4) and (3.7) are equivalent.

(ii) For  $p < 0$ , or  $0 < p < 1$ , we similarly obtain the reverse of (3.7), that is equivalent to the reverses of (3.3) and (3.4).  $\square$

**Lemma 3.4.** *If*

$$\rho > 0, \tilde{\sigma} > \eta > 0, \zeta\left(\frac{\tilde{\sigma}}{\eta}\right) = \sum_{k=1}^{\infty} \frac{1}{k^{\tilde{\sigma}/\eta}},$$

where  $\zeta(\cdot)$  stands for the Riemann zeta function, then we have

$$\begin{aligned} \omega_\delta(\tilde{\sigma}, n) &= K_2(\tilde{\sigma}) := \frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0-1} \Gamma(\frac{i_0}{\alpha})} k(\tilde{\sigma}), \\ k(\tilde{\sigma}) &:= \frac{2}{\eta \rho^{\tilde{\sigma}/\eta}} \Gamma\left(\frac{\tilde{\sigma}}{\eta}\right) \left(1 - \frac{1}{2^{\tilde{\sigma}/\eta}}\right) \zeta\left(\frac{\tilde{\sigma}}{\eta}\right) \in \mathbf{R}_+ (n \in \mathbf{N}^{j_0}). \end{aligned} \quad (3.8)$$

Moreover, there exists a  $\tilde{\theta} \in (0, 1)$ , such that  $(1 - \tilde{\theta})\tilde{\sigma} > \eta$ .

If  $0 < \beta \leq 1, \eta \leq 1, 0 < \eta < \tilde{\sigma} < j_0$ , then we obtain

$$K_1(\tilde{\sigma})(1 - \theta_{\tilde{\sigma}}(\|x\|_\alpha^\delta)) < \varpi_\delta(\tilde{\sigma}, x) < K_1(\tilde{\sigma})(x \in \mathbf{R}_+^{i_0}). \quad (3.9)$$

where,

$$K_1(\tilde{\sigma}) := \frac{\Gamma^{j_0}(\frac{1}{\beta})}{\beta^{j_0-1} \Gamma(\frac{j_0}{\beta})} k(\tilde{\sigma}) \in \mathbf{R}_+, \quad (3.10)$$

$$\theta_{\tilde{\sigma}}(\|x\|_\alpha^\delta) := \frac{1}{k(\tilde{\sigma})} \int_{\|x\|_\alpha^\delta j_0^{-1/\beta}}^\infty \csc h\left(\frac{\rho}{t^\eta}\right) t^{-\tilde{\sigma}-1} dt = O\left(\frac{1}{\|x\|_\alpha^{\delta\tilde{\theta}\tilde{\sigma}}}\right) \in (0, 1). \quad (3.11)$$

*Proof.* By (2.4), since  $\delta = \pm 1$ , for

$$k_0(x, y) = \csc h\left(\frac{\rho y^\eta}{x^\eta}\right), k_0(|x|_\alpha^\delta, |y - \tau|_\beta) = \csc h\left(\frac{\rho |n - \tau|_\beta^\eta}{|x|_\alpha^{\delta\eta}}\right),$$

we find

$$\begin{aligned} \omega_\delta(\tilde{\sigma}, n) &= ||n - \tau||_\beta^{\tilde{\sigma}} \int_{\mathbf{R}_+^{i_0}} k_0 \left( M^\delta \left( \sum_{i=1}^{i_0} \left( \frac{x_i}{M} \right)^\alpha \right)^{\frac{\delta}{\alpha}}, ||n - \tau||_\beta \right) \\ &\quad \times \frac{1}{M^{\delta\tilde{\sigma}+i_0} \left( \sum_{i=1}^{i_0} \left( \frac{x_i}{M} \right)^\alpha \right)^{\frac{\delta\tilde{\sigma}+i_0}{\alpha}}} dx_1 \cdots dx_{i_0} \\ &= \lim_{M \rightarrow \infty} \frac{M^{i_0} \Gamma(i_0)}{\alpha^{i_0} \Gamma(\frac{i_0}{\alpha})} ||n - \tau||_\beta^{\tilde{\sigma}} \int_0^1 \frac{k_0(M^\delta u^{\frac{\delta}{\alpha}}, ||n - \tau||_\beta)}{M^{\delta\tilde{\sigma}+i_0} u^{\frac{\delta\tilde{\sigma}+i_0}{\alpha}}} u^{\frac{i_0}{\alpha}-1} du \\ &= \frac{\Gamma(i_0)}{\alpha^{i_0-1} \Gamma(\frac{i_0}{\alpha})} \int_0^\infty k_0(v, 1) v^{-\tilde{\sigma}-1} dv \\ &= \frac{\Gamma(i_0)}{\alpha^{i_0-1} \Gamma(\frac{i_0}{\alpha})} \int_0^\infty \csc h\left(\frac{\rho}{v^\eta}\right) v^{-\tilde{\sigma}-1} dv. \end{aligned}$$

Thus we obtain

$$\begin{aligned} \int_0^\infty \csc h\left(\frac{\rho}{v^\eta}\right) v^{-\tilde{\sigma}-1} dv &= \int_0^\infty \frac{2}{e^{\frac{\rho}{v^\eta}} - e^{-\frac{\rho}{v^\eta}}} v^{-\tilde{\sigma}-1} dv \\ &\stackrel{u=\frac{\rho}{v^\eta}}{=} \frac{2}{\eta \rho^{\tilde{\sigma}/\eta}} \int_0^\infty \frac{u^{(\tilde{\sigma}/\eta)-1} du}{e^u (1 - e^{-2u})} = \frac{2}{\eta \rho^{\tilde{\sigma}/\eta}} \int_0^\infty \sum_{k=0}^\infty e^{-(2k+1)u} u^{(\tilde{\sigma}/\eta)-1} du \\ &= \frac{2}{\eta \rho^{\tilde{\sigma}/\eta}} \sum_{k=0}^\infty \int_0^\infty e^{-(2k+1)u} u^{(\tilde{\sigma}/\eta)-1} du \\ &\stackrel{t=(2k+1)u}{=} \frac{2}{\eta \rho^{\tilde{\sigma}/\eta}} \sum_{k=0}^\infty \frac{1}{(2k+1)^{\tilde{\sigma}/\eta}} \int_0^\infty e^{-t} t^{(\tilde{\sigma}/\eta)-1} dt \\ &= \frac{2}{\eta \rho^{\tilde{\sigma}/\eta}} \Gamma\left(\frac{\tilde{\sigma}}{\eta}\right) \left( \sum_{k=1}^\infty \frac{1}{k^{\tilde{\sigma}/\eta}} - \sum_{k=1}^\infty \frac{1}{(2k)^{\tilde{\sigma}/\eta}} \right) \\ &= \frac{2}{\eta \rho^{\tilde{\sigma}/\eta}} \Gamma\left(\frac{\tilde{\sigma}}{\eta}\right) \left(1 - \frac{1}{2^{\tilde{\sigma}/\eta}}\right) \zeta\left(\frac{\tilde{\sigma}}{\eta}\right). \end{aligned}$$

Hence, (3.8) follows. Moreover, since  $0 < \beta \leq 1$ ,  $\eta \leq 1$ ,  $\tilde{\sigma} < j_0$ , we get

$$\frac{\partial}{\partial y} \left( \csc h\left(\frac{\rho y^\eta}{x^\eta}\right) y^{\tilde{\sigma}-j_0} \right) < 0, \quad \frac{\partial^2}{\partial y^2} \left( \csc h\left(\frac{\rho y^\eta}{x^\eta}\right) y^{\tilde{\sigma}-j_0} \right) > 0,$$

and by (2.1) and (2.4), we obtain

$$\begin{aligned} \varpi_\delta(\tilde{\sigma}, x) &< ||x|_\alpha|^{-\delta\tilde{\sigma}} \int_{\{y \in \mathbf{R}_+^{j_0}; y_i \geq \frac{1}{2}\}} k_0(|x|_\alpha^\delta, |y - \tau|_\beta) \frac{dy}{||y - \tau||_\beta^{j_0-\tilde{\sigma}}} \\ &= ||x|_\alpha|^{-\delta\tilde{\sigma}} \int_{\{u \in \mathbf{R}_+^{j_0}; u_i \geq \frac{1}{2} - \tau_i\}} k_0(|x|_\alpha^\delta, |u|_\beta) \frac{du}{||u|_\beta^{j_0-\tilde{\sigma}}} \end{aligned}$$

$$\begin{aligned}
&\leq ||x||_{\alpha}^{-\delta\tilde{\sigma}} \int_{\mathbf{R}_+^{j_0}} k_0(||x||_{\alpha}^{\delta}, ||u||_{\beta}) \frac{du}{||u||_{\beta}^{j_0-\tilde{\sigma}}} \\
&= \frac{\Gamma^{j_0}(\frac{1}{\beta})}{\beta^{j_0-1}\Gamma(\frac{j_0}{\beta})} \int_0^{\infty} k_0(v, 1) v^{-\tilde{\sigma}-1} dv = K_1(\tilde{\sigma}).
\end{aligned}$$

By (2.1), (2.7) and (3.8), we find

$$\begin{aligned}
w_{\delta}(\tilde{\sigma}, x) &> ||x||_{\alpha}^{-\delta\tilde{\sigma}} \int_{\{y \in \mathbf{R}_+^{j_0}, y_i \geq 1+\tau_i\}} k_0(||x||_{\alpha}^{\delta}, ||y-\tau||_{\beta}) \frac{dy}{||y-\tau||_{\beta}^{j_0-\tilde{\sigma}}} \\
&= ||x||_{\alpha}^{-\delta\tilde{\sigma}} \int_{\{u \in \mathbf{R}_+^{j_0}, u_i \geq 1\}} k_0(||x||_{\alpha}^{\delta}, ||u||_{\beta}) \frac{du}{||u||_{\beta}^{j_0-\tilde{\sigma}}} \\
&= ||x||_{\alpha}^{-\delta\tilde{\sigma}} \lim_{M \rightarrow \infty} \frac{M^{j_0}\Gamma^{j_0}(\frac{1}{\beta})}{\beta^{j_0-1}\Gamma(\frac{j_0}{\beta})} \int_{\frac{j_0}{M^{\beta}}}^1 \frac{k_0(||x||_{\alpha}^{\delta}, Mv^{\frac{1}{\beta}})}{(Mv^{\frac{1}{\beta}})^{j_0-\tilde{\sigma}}} v^{\frac{j_0}{\beta}-1} dv \\
&= \frac{\Gamma^{j_0}(\frac{1}{\beta})}{\beta^{j_0-1}\Gamma(\frac{j_0}{\beta})} \int_0^{\frac{||x||_{\alpha}^{\delta} j_0^{-1/\beta}}{t}} k_0(t, 1) t^{-\tilde{\sigma}-1} dt \\
&= \frac{\Gamma^{j_0}(\frac{1}{\beta})}{\beta^{j_0-1}\Gamma(\frac{j_0}{\beta})} k(\tilde{\sigma}) [1 - \theta_{\tilde{\sigma}}(||x||_{\alpha}^{\delta})] > 0, \\
0 < \theta_{\tilde{\sigma}}(||x||_{\alpha}^{\delta}) &= \frac{1}{k(\tilde{\sigma})} \int_{\frac{||x||_{\alpha}^{\delta} j_0^{-1/\beta}}{t}}^{\infty} \csc h(\frac{\rho}{t^{\eta}}) t^{-\tilde{\sigma}-1} dt.
\end{aligned}$$

Since we have

$$t^{(\tilde{\theta}-1)\tilde{\sigma}} \csc h(\frac{\rho}{t^{\eta}}) \rightarrow 0 \quad (t \rightarrow 0^+, \text{ or } t \rightarrow \infty),$$

there exists  $L > 0$ , such that

$$t^{(\tilde{\theta}-1)\tilde{\sigma}} \csc h(\frac{\rho}{t^{\eta}}) \leq L \quad (t \in \mathbf{R}_+)$$

and then

$$\theta_{\tilde{\sigma}}(||x||_{\alpha}^{\delta}) \leq \frac{L}{k(\tilde{\sigma})} \int_{\frac{||x||_{\alpha}^{\delta} j_0^{-1/\beta}}{t}}^{\infty} t^{-\tilde{\theta}\tilde{\sigma}-1} dt = \frac{L j_0^{(\tilde{\theta}\tilde{\sigma})/\beta}}{k(\tilde{\sigma}) \tilde{\theta}\tilde{\sigma}} \frac{1}{||x||_{\alpha}^{\delta\tilde{\theta}\tilde{\sigma}}}.$$

Hence (3.9), (3.10) and (3.11) follow.  $\square$

**Note.** The following references [36]–[43] provide an extensive theory and applications of Analytic Number Theory related to the Riemann zeta function that offers a source of study for further research on Hilbert – type inequalities.

Setting

$$\Phi_{\delta}(x) := ||x||_{\alpha}^{p(i_0+\delta\sigma)-i_0}, \Psi(n) := ||n-\tau||_{\beta}^{q(j_0-\sigma)-j_0},$$

$$\tilde{\Phi}_{\delta}(x) := (1 - \theta_{\sigma}(||x||_{\alpha}^{\delta})) ||x||_{\alpha}^{p(i_0+\delta\sigma)-i_0}, \quad (\theta_{\lambda_l}(||x||_{\alpha}^{\delta}) \in (0, 1); x \in \mathbf{R}_+^{i_0}),$$

by Lemma 3.3 and Lemma 3.4, we obtain

**Theorem 3.5.** Suppose that

$$\alpha > 0, 0 < \beta \leq 1, \tau = (\tau_1, \dots, \tau_{j_0}) \in [0, \frac{1}{2}]^{j_0}, \delta \in \{-1, 1\}, \rho > 0, \eta \leq 1, 0 < \eta < \sigma < j_0.$$

If  $p > 1$ ,  $f(x) = f(x_1, \dots, x_{i_0}) \geq 0, a_n = a_{(n_1, \dots, n_{j_0})} \geq 0$ ,

$$0 < \|f\|_{p, \Phi_\delta} = \left( \int_{\mathbf{R}_+^{i_0}} \Phi_\delta(x) f^p(x) dx \right)^{\frac{1}{p}} < \infty,$$

$$0 < \|a\|_{q, \Psi} = \left( \sum_n \Psi(n) a_n^q \right)^{\frac{1}{q}} < \infty,$$

then we have the following equivalent inequalities with the best possible constant factor  $K(\sigma)$ :

$$I = \sum_n \int_{\mathbf{R}_+^{i_0}} \csc h \left( \frac{\rho |n - \tau|_\beta^\eta}{||x||_\alpha^{\delta\eta}} \right) a_n f(x) dx < K(\sigma) \|f\|_{p, \Phi_\delta} \|a\|_{q, \Psi}, \quad (3.12)$$

$$J := \left( \sum_n |n - \tau|_\beta^{p\sigma - i_0} \left( \int_{\mathbf{R}_+^{i_0}} \csc h \left( \frac{\rho |n - \tau|_\beta^\eta}{||x||_\alpha^{\delta\eta}} \right) f(x) dx \right)^p \right)^{\frac{1}{p}} < K(\sigma) \|f\|_{p, \Phi_\delta}, \quad (3.13)$$

$$H := \left( \int_{\mathbf{R}_+^{i_0}} ||x||_\alpha^{-\delta q \sigma - i_0} \left( \sum_n \csc h \left( \frac{\rho |n - \tau|_\beta^\eta}{||x||_\alpha^{\delta\eta}} \right) a_n \right)^q dx \right)^{\frac{1}{q}} < K(\sigma) \|a\|_{q, \Psi}, \quad (3.14)$$

where

$$k(\sigma) = \frac{2}{\eta \rho^{\sigma/\eta}} \Gamma(\frac{\sigma}{\eta}) (1 - \frac{1}{2^{\sigma/\eta}}) \xi(\frac{\sigma}{\eta}),$$

$$\begin{aligned} K(\sigma) &:= (K_1(\sigma))^{\frac{1}{p}} (K_2(\sigma))^{\frac{1}{q}} \\ &= \left( \frac{\Gamma^{j_0}(\frac{1}{\beta})}{\beta^{j_0-1} \Gamma(\frac{j_0}{\beta})} \right)^{\frac{1}{p}} \left( \frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0-1} \Gamma(\frac{i_0}{\alpha})} \right)^{\frac{1}{q}} k(\sigma). \end{aligned}$$

In particular, for  $i_0 = j_0 = 1, \tau \in [0, \frac{1}{2}], \rho > 0, 0 < \eta < \sigma < 1$ ,

$$\varphi_\delta(x) := x^{p(1+\delta\sigma)-1}, \psi(n) := (n - \tau)^{q(1-\sigma)-1},$$

we have the following equivalent inequalities with the best possible constant factor  $k(\sigma)$ :

$$\sum_{n=1}^{\infty} \int_0^{\infty} \csc h \left( \frac{\rho (n - \tau)^\eta}{x^{\delta\eta}} \right) a_n f(x) dx < k(\sigma) \|f\|_{p, \varphi_\delta} \|a\|_{q, \psi}, \quad (3.15)$$

$$\left( \sum_{n=1}^{\infty} (n - \tau)^{p\sigma - 1} \left( \int_0^{\infty} \csc h \left( \frac{\rho (n - \tau)^\eta}{x^{\delta\eta}} \right) f(x) dx \right)^p \right)^{\frac{1}{p}} < k(\sigma) \|f\|_{p, \varphi_\delta}, \quad (3.16)$$

$$\left( \int_0^{\infty} x^{-\delta q \sigma - 1} \left( \sum_{n=1}^{\infty} \csc h \left( \frac{\rho (n - \tau)^\eta}{x^{\delta\eta}} \right) a_n \right)^q dx \right)^{\frac{1}{q}} < k(\sigma) \|a\|_{q, \psi}. \quad (3.17)$$

*Proof.* By Lemma 3.2, Lemma 3.3 and Lemma 3.4, we get equivalent inequalities (3.12), (3.13) and (3.14). By Hölder's inequality, we have

$$I \leq J \left( \sum_n \|n - \tau\|_{\beta}^{q(j_0 - \sigma) - j_0} a_n^q \right)^{\frac{1}{q}}, \quad (3.18)$$

$$I \leq \left( \int_{\mathbf{R}_+^{i_0}} \|x\|_{\alpha}^{p(i_0 + \delta\sigma) - i_0} f^p(x) dx \right)^{\frac{1}{p}} H. \quad (3.19)$$

For  $\delta_0 = \min\{\eta, \sigma - \eta, j_0 - \sigma\}, 0 < \varepsilon < q\delta_0$ , we set  $\tilde{f}(x), \tilde{a}_n$  as follows:

$$\begin{aligned} \tilde{f}(x) &:= \begin{cases} 0, & 0 < \|x\|_{\alpha}^{\delta} < 1, \\ \|x\|_{\alpha}^{-\delta\sigma - \frac{\delta\eta}{p} - i_0}, & \|x\|_{\alpha}^{\delta} \geq 1, \end{cases} \\ \tilde{a}_n &:= \|n - \tau\|_{\beta}^{(\sigma - \frac{\varepsilon}{q}) - j_0}, n \in \mathbf{N}^{j_0}. \end{aligned}$$

Then for  $0 < \tilde{\sigma} = \sigma - \frac{\varepsilon}{q} (< j_0)$ , in view of (2.8), (2.9) and (3.11), we find

$$\begin{aligned} \|\tilde{f}\|_{p, \Phi_{\delta}} \|\tilde{a}\|_{q, \Psi} &= \left( \int_{\{x \in \mathbf{R}_+^{i_0}; \|x\|_{\alpha}^{\delta} \geq 1\}} \|x\|_{\alpha}^{-i_0 - \delta\varepsilon} dx \right)^{\frac{1}{p}} \left( \sum_n \|n - \tau\|_{\beta}^{-j_0 - \varepsilon} \right)^{\frac{1}{q}} \\ &= \frac{1}{\varepsilon} \left( \frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0 - 1} i_0^{\delta\varepsilon/\alpha} \Gamma(\frac{i_0}{\alpha})} \right)^{\frac{1}{p}} \left( \frac{\Gamma^{j_0}(\frac{1}{\beta})}{J_0^{\varepsilon/\beta} \beta^{j_0 - 1} \Gamma(\frac{j_0}{\beta})} + \varepsilon O(1) \right)^{\frac{1}{q}}, \\ \tilde{I} &:= \int_{\mathbf{R}_+^{i_0}} \sum_n \csc h \left( \frac{\rho \|n - \tau\|_{\beta}^{\eta}}{\|x\|_{\alpha}^{\delta\eta}} \right) \tilde{a}_n \tilde{f}(x) dx \\ &= \int_{\{x \in \mathbf{R}_+^{i_0}; \|x\|_{\alpha}^{\delta} \geq 1\}} \|x\|_{\alpha}^{-i_0 - \delta\varepsilon} \varpi_{\delta}(\tilde{\sigma}, x) dx \\ &\geq K_1(\tilde{\sigma}) \int_{\{x \in \mathbf{R}_+^{i_0}; \|x\|_{\alpha}^{\delta} \geq 1\}} \|x\|_{\alpha}^{-i_0 - \delta\varepsilon} (1 - O(\frac{1}{\|x\|_{\alpha}^{\delta\tilde{\sigma}}})) dx \\ &= \frac{1}{\varepsilon} K_1(\tilde{\sigma}) \left( \frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0 - 1} \Gamma(\frac{i_0}{\alpha})} - \varepsilon O_{\tilde{\sigma}}(1) \right). \end{aligned}$$

If there exists a constant  $K \leq K(\sigma)$ , such that (3.12) is valid when replacing  $K(\sigma)$  by  $K$ , then in particular, by the above results, we have

$$\begin{aligned} K_1(\tilde{\sigma}) \left( \frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0 - 1} i_0^{\delta\varepsilon/\alpha} \Gamma(\frac{i_0}{\alpha})} - \varepsilon O_{\tilde{\sigma}}(1) \right) &\leq \varepsilon \tilde{I} < \varepsilon K \|\tilde{f}\|_{p, \Phi} \|\tilde{a}\|_{q, \Psi} \\ &= K \left( \frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0 - 1} \Gamma(\frac{i_0}{\alpha})} \right)^{\frac{1}{p}} \left( \frac{\Gamma^{j_0}(\frac{1}{\beta})}{J_0^{\varepsilon/\beta} \beta^{j_0 - 1} \Gamma(\frac{j_0}{\beta})} + \varepsilon O(1) \right)^{\frac{1}{q}}, \end{aligned}$$

and thus  $K(\sigma) \leq K (\varepsilon \rightarrow 0^+)$ . Hence  $K = K(\sigma)$  is the best possible constant factor of (3.12).

By the equivalency, we can prove that the constant factor  $K(\sigma)$  in (3.13) and (3.14) is the best possible. Otherwise, we would reach a contradiction by (3.18) and (3.19) that the constant factor  $K(\sigma)$  in (3.12) is not the best possible.  $\square$

**Theorem 3.6.** *By the assumptions of Theorem 3.5, if  $p < 0$  ( $0 < q < 1$ ),*

$$f(x) = f(x_1, \dots, x_{i_0}) \geq 0, a_n = a_{(n_1, \dots, n_{j_0})} \geq 0, 0 < \|f\|_{p, \Phi_\delta} < \infty, 0 < \|a\|_{q, \Psi} < \infty,$$

*then we have the equivalent reverses of (3.12), (3.13) and (3.14) with the best possible constant factor  $K(\sigma)$ :*

*In particular, for*

$$i_0 = j_0 = 1, \tau \in [0, \frac{1}{2}], \rho > 0, 0 < \eta < \sigma < 1,$$

$\varphi_\delta(x)$  and  $\psi(n)$  as indicated in Theorem 3.5, we have the equivalent reverse of (3.15), (3.16) and (3.17) with the best possible constant factor  $k(\sigma)$ .

*Proof.* We only prove that the constant factor  $K(\sigma)$  in the reverse of (3.12) is the best possible. The rest is omitted. For  $0 < \varepsilon < q\delta_0$ ,  $\tilde{\sigma} = \sigma - \frac{\varepsilon}{q}$ , we set  $\tilde{f}(x), \tilde{a}_n$  as in Theorem 3.5. If there exists a constant  $K \geq K(\sigma)$ , such that the reverse inequality of (3.12) is valid when replacing  $K(\sigma)$  by  $K$  then, we obtain

$$\begin{aligned} & K \left( \frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0-1} i_0^{\delta\varepsilon/\alpha} \Gamma(\frac{i_0}{\alpha})} \right)^{\frac{1}{p}} \left( \frac{\Gamma^{j_0}(\frac{1}{\beta})}{J_0^{\varepsilon/\beta} \beta^{j_0-1} \Gamma(\frac{j_0}{\beta})} + \varepsilon O(1) \right)^{\frac{1}{q}} \\ &= \varepsilon K \|\tilde{f}\|_{p, \Phi} \|\tilde{a}\|_{q, \Psi} < \varepsilon \tilde{I} < \varepsilon \int_{\{x \in \mathbf{R}_+^{i_0}; \|x\|_\alpha^\delta \geq 1\}} \|x\|_\alpha^{-i_0-\delta\varepsilon} \varpi_\delta(\tilde{\sigma}, x) dx \\ &< \varepsilon K_2(\tilde{\sigma}) \int_{\{x \in \mathbf{R}_+^{i_0}; \|x\|_\alpha^\delta \geq 1\}} \|x\|_\alpha^{-i_0-\delta\varepsilon} dx = K_2(\tilde{\sigma}) \frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0-1} \Gamma(\frac{i_0}{\alpha})}, \end{aligned}$$

and thus  $K \leq K(\sigma)$  ( $\varepsilon \rightarrow 0^+$ ). Hence  $K = K(\sigma)$  is the best possible constant factor of the reverse inequality of (3.12).  $\square$

**Theorem 3.7.** *By the assumptions of Theorem 3.5, if  $0 < p < 1$  ( $q < 0$ ),*

$$f(x) = f(x_1, \dots, x_{i_0}) \geq 0, a_n = a_{(n_1, \dots, n_{j_0})} \geq 0, 0 < \|f\|_{p, \tilde{\Phi}_\delta} < \infty, 0 < \|a\|_{q, \Psi} < \infty,$$

*then we have the following equivalent inequalities with the best possible constant factor  $K(\sigma)$ :*

$$\sum_n \int_{\mathbf{R}_+^{i_0}} \csc h \left( \frac{\rho \|n - \tau\|_\beta^\eta}{\|x\|_\alpha^\delta} \right) a_n f(x) dx > K(\sigma) \|f\|_{p, \tilde{\Phi}_\delta} \|a\|_{q, \Psi}, \quad (3.20)$$

$$\left( \sum_n \|n - \tau\|_\beta^{p\sigma-j_0} \left( \int_{\mathbf{R}_+^{i_0}} \csc h \left( \frac{\rho \|n - \tau\|_\beta^\eta}{\|x\|_\alpha^\delta} \right) f(x) dx \right)^p \right)^{\frac{1}{p}} > K(\sigma) \|f\|_{p, \tilde{\Phi}_\delta}, \quad (3.21)$$

$$\left( \int_{\mathbf{R}_+^{i_0}} \frac{\|x\|_\alpha^{-\delta q \sigma - i_0}}{(1 - \theta_\sigma(\|x\|_\alpha^\delta))^{q-1}} \left( \sum_n \csc h \left( \frac{\rho \|n - \tau\|_\beta^\eta}{\|x\|_\alpha^\delta} \right) a_n \right)^q dx \right)^{\frac{1}{q}} > K(\sigma) \|a\|_{q, \Psi}. \quad (3.22)$$

In particular, for  $i_0 = j_0 = 1, \tau \in [0, \frac{1}{2}], \rho > 0, 0 < \eta < \sigma < 1$ ,  $\varphi_\delta(x)$  and  $\psi(n)$  as defined in Theorem 3.5,

$$\tilde{\varphi}_\delta(x) = (1 - \theta_\sigma(x^\delta))x^{p(1+\delta\sigma)-1} \quad (\theta_\sigma(x^\delta) \in (0, 1); x \in \mathbf{R}_+),$$

we have the following equivalent inequalities with the best possible constant factor  $k(\sigma)$ :

$$\sum_{n=1}^{\infty} \int_0^{\infty} \csc h \left( \frac{\rho(n-\tau)^\eta}{x^{\delta\eta}} \right) a_n f(x) dx > k(\sigma) \|f\|_{p,\varphi_\delta} \|a\|_{q,\psi}, \quad (3.23)$$

$$\left( \sum_{n=1}^{\infty} (n-\tau)^{p\sigma-1} \left( \int_0^{\infty} \csc h \left( \frac{\rho(n-\tau)^\eta}{x^{\delta\eta}} \right) f(x) dx \right)^p \right)^{\frac{1}{p}} > k(\sigma) \|f\|_{p,\tilde{\varphi}_\delta}, \quad (3.24)$$

$$\left( \int_0^{\infty} \frac{x^{-\delta q \sigma - 1}}{(1 - \theta_\sigma(x^\delta))^{q-1}} \left( \sum_{n=1}^{\infty} \csc h \left( \frac{\rho(n-\tau)^\eta}{x^{\delta\eta}} \right) a_n \right)^q dx \right)^{\frac{1}{q}} > k(\sigma) \|a\|_{q,\psi}. \quad (3.25)$$

*Proof.* We only prove that the constant factor  $K(\sigma)$  in (3.20) is the best possible. The rest is omitted. For  $0 < \varepsilon < |q|\delta_0$ ,  $\tilde{\sigma} = \sigma - \frac{\varepsilon}{q} (< j_0)$ , we set  $\tilde{f}(x), \tilde{a}_n$  as in Theorem 3.5. Then in view of (2.5), (2.6) and (2.9), we get

$$\begin{aligned} \|\tilde{f}\|_{p,\tilde{\varphi}_\delta} \|\tilde{a}\|_{q,\Psi} &= \left( \int_{\{x \in \mathbf{R}_+^i; |x|_\alpha^\delta \geq 1\}} |x|_\alpha^{-i_0 - \delta\varepsilon} (1 - O(|x|_\alpha^{-\delta\theta\sigma})) dx \right)^{\frac{1}{p}} \\ &\times \left( \sum_n \|n - \tau\|_\beta^{-j_0 - \varepsilon} \right)^{\frac{1}{q}} \\ &= \left( \frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0-1} i_0^{\delta\varepsilon/\alpha} \Gamma(\frac{i_0}{\alpha})} - \varepsilon \tilde{O}(1) \right)^{\frac{1}{p}} \left( \frac{\Gamma^{j_0}(\frac{1}{\beta})}{J_0^{\varepsilon/\beta} \beta^{j_0-1} \Gamma(\frac{j_0}{\beta})} + \varepsilon O(1) \right)^{\frac{1}{q}}. \end{aligned}$$

If there exists a constant  $K \geq K(\sigma)$ , such that (3.20) is valid when replacing  $K(\sigma)$  by  $K$ , by (3.9) and the above results, we have

$$\begin{aligned} K \left( \frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0-1} i_0^{\delta\varepsilon/\alpha} \Gamma(\frac{i_0}{\alpha})} - \varepsilon \tilde{O}(1) \right)^{\frac{1}{p}} \left( \frac{\Gamma^{j_0}(\frac{1}{\beta})}{J_0^{\varepsilon/\beta} \beta^{j_0-1} \Gamma(\frac{j_0}{\beta})} + \varepsilon O(1) \right)^{\frac{1}{q}} \\ = \varepsilon K \|\tilde{f}\|_{p,\tilde{\varphi}_\delta} \|\tilde{a}\|_{q,\Psi} < \varepsilon \tilde{I} < \varepsilon \int_{\{x \in \mathbf{R}_+^i; |x|_\alpha^\delta \geq 1\}} |x|_\alpha^{-i_0 - \delta\varepsilon} \varpi_\delta(\tilde{\sigma}, x) dx \\ < \varepsilon K_2(\tilde{\sigma}) \int_{\{x \in \mathbf{R}_+^i; |x|_\alpha^\delta \geq 1\}} |x|_\alpha^{-i_0 - \delta\varepsilon} dx = K_2(\tilde{\sigma}) \frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0-1} \Gamma(\frac{i_0}{\alpha})}, \end{aligned}$$

and then  $K \leq K(\sigma)$  ( $\varepsilon \rightarrow 0^+$ ). Hence  $K = K(\sigma)$  is the best possible constant factor of (3.20).  $\square$

For  $p > 1$ , we set

$$\Phi_\delta(x) = ||x||_\alpha^{p(i_0 + \delta\sigma) - i_0} (x \in \mathbf{R}_+^{i_0}), \Psi(n) = ||n - \tau||_\beta^{q(j_0 - \sigma) - j_0} (n \in \mathbf{N}^{j_0})$$

and hence

$$[\Psi(n)]^{1-p} = ||n - \tau||_\beta^{p\sigma - j_0}, [\Phi_\delta(x)]^{1-q} = ||x||_\alpha^{-\delta q\sigma - j_0}.$$

We define two real weight normal spaces  $L_{p,\Phi_\delta}(\mathbf{R}_+^{i_0})$  and  $l_{q,\Psi}$  as follows:

$$\begin{aligned} L_{p,\Phi_\delta}(\mathbf{R}_+^{i_0}) &: = \left( f; ||f||_{p,\Phi_\delta} = \left( \int_{\mathbf{R}_+^{i_0}} \Phi_\delta(x) |f(x)|^p dx \right)^{\frac{1}{p}} < \infty \right), \\ l_{q,\Psi} &: = \left( a = \{a_n\}; ||a||_{q,\Psi} = \left( \sum_n \Psi(n) |a_n|^q \right)^{\frac{1}{q}} < \infty \right). \end{aligned}$$

By the assumptions of Theorem 3.5, in view of

$$J < K(\sigma) ||f||_{p,\Phi_\delta}, \quad H < K(\sigma) ||a||_{q,\Psi},$$

we formulate the following definition:

**Definition 3.8.** We define a first kind of multidimensional half-discrete Hilbert-type operator  $T_1 : L_{p,\Phi_\delta}(\mathbf{R}_+^{i_0}) \rightarrow l_{p,\Psi^{1-p}}$  as follows: For  $f \in L_{p,\Phi_\delta}(\mathbf{R}_+^{i_0})$ , there exists a unique representation  $T_1 f \in l_{p,\Psi^{1-p}}$ , satisfying

$$(T_1 f)(n) := \int_{\mathbf{R}_+^{i_0}} \csc h \left( \frac{\rho(n - \tau)^\eta}{x^{\delta\eta}} \right) f(x) dx \quad (n \in \mathbf{N}^{j_0}). \quad (3.26)$$

For  $a \in l_{q,\Psi}$ , we define the following formal inner product of  $T_1 f$  and  $a$  as follows:

$$(T_1 f, a) := \sum_n a_n \int_{\mathbf{R}_+^{i_0}} \csc h \left( \frac{\rho(n - \tau)^\eta}{x^{\delta\eta}} \right) a_n f(x) dx. \quad (3.27)$$

We define a second kind of multidimensional half-discrete Hilbert-type operator

$$T_2 : l_{q,\Psi} \rightarrow L_{q,\Phi_\delta^{1-q}}(\mathbf{R}_+^{i_0})$$

as follows: For  $a \in l_{q,\Psi}$ , there exists a unique representation

$$T_2 a \in L_{q,\Phi_\delta^{1-q}}(\mathbf{R}_+^{i_0}),$$

satisfying

$$(T_2 a)(x) := \sum_n \csc h \left( \frac{\rho(n - \tau)^\eta}{x^{\delta\eta}} \right) a_n \quad (x \in \mathbf{R}_+^{i_0}). \quad (3.28)$$

For  $f \in L_{p,\Phi_\delta}(\mathbf{R}_+^{i_0})$ , we define the following formal inner product of  $f$  and  $T_2 a$  as follows:

$$(f, T_2 a) := \int_{\mathbf{R}_+^{i_0}} \sum_n \csc h \left( \frac{\rho(n - \tau)^\eta}{x^{\delta\eta}} \right) a_n f(x) dx. \quad (3.29)$$

Then by Theorem 3.5, for

$$0 < \|f\|_{p,\Phi_\delta}, \quad \|a\|_{q,\Psi} < \infty,$$

we have the following equivalent inequalities:

$$(T_1 f, a) = (f, T_2 a) < K(\sigma) \|f\|_{p,\Phi_\delta} \|a\|_{q,\Psi}, \quad (3.30)$$

$$\|T_1 f\|_{p,\Psi^{1-p}} < K(\sigma) \|f\|_{p,\Phi_\delta}, \quad (3.31)$$

$$\|T_2 a\|_{q,\Phi_\delta^{1-q}} < K(\sigma) \|a\|_{q,\Psi}. \quad (3.32)$$

It follows that  $T_1$  and  $T_2$  are bounded with

$$\begin{aligned} \|T_1\| &:= \sup_{f(\neq 0) \in L_{p,\Phi_\delta}(\mathbf{R}_+^{i_0})} \frac{\|T_1 f\|_{p,\Psi^{1-p}}}{\|f\|_{p,\Phi_\delta}} \leq K(\sigma), \\ \|T_2\| &:= \sup_{a(\neq 0) \in l_{q,\Psi}} \frac{\|T_2 a\|_{q,\Phi_\delta^{1-q}}}{\|a\|_{q,\Psi}} \leq K(\sigma). \end{aligned}$$

Since the constant factor  $K(\sigma)$  in (3.31) and (3.32) is the best possible, we obtain

$$\|T_1\| = \|T_2\| = K(\sigma) = \left( \frac{\Gamma^{j_0}(\frac{1}{\beta})}{\beta^{j_0-1} \Gamma(\frac{j_0}{\beta})} \right)^{\frac{1}{p}} \left( \frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0-1} \Gamma(\frac{i_0}{\alpha})} \right)^{\frac{1}{q}} k(\sigma). \quad (3.33)$$

## 4 A Corollary on $\delta = -1$

**Corollary 4.1.** *By the assumptions of Theorems 3.5–3.7, for  $\delta = -1$ ,*

$$\Phi(x) := \|x\|_\alpha^{p(i_0-\sigma)-i_0} \text{ and } \tilde{\Phi}(x) = (1 - \theta_\sigma(\|x\|_\alpha^{-1})) \Phi(x).$$

(i) If  $p > 1$ , then we get the following equivalent inequalities with the best possible constant factor  $K(\sigma)$ :

$$\sum_n \int_{\mathbf{R}_+^{i_0}} \csc h \left( \rho \|x\|_\alpha^{\eta} \|n - \tau\|_\beta^{\eta} \right) a_n f(x) dx < K(\sigma) \|f\|_{p,\Phi} \|a\|_{q,\Psi}, \quad (4.1)$$

$$\left( \sum_n \|n - \tau\|_\beta^{p\sigma - j_0} \left( \int_{\mathbf{R}_+^{i_0}} \csc h \left( \rho \|x\|_\alpha^{\eta} \|n - \tau\|_\beta^{\eta} \right) f(x) dx \right)^p \right)^{\frac{1}{p}} < K(\sigma) \|f\|_{p,\Phi}, \quad (4.2)$$

$$\left( \int_{\mathbf{R}_+^{i_0}} \|x\|_\alpha^{q\sigma - i_0} \left( \sum_n \csc h \left( \rho \|x\|_\alpha^{\eta} \|n - \tau\|_\beta^{\eta} \right) a_n \right)^q dx \right)^{\frac{1}{q}} < K(\sigma) \|a\|_{q,\Psi}; \quad (4.3)$$

(ii) if  $p < 0$  ( $0 < q < 1$ ), then we have the equivalent reverses of (4.1), (4.2) and (4.3) with the same best possible constant factor  $K(\sigma)$ ;

(iii) if  $0 < p < 1$  ( $q < 0$ ), we obtain the following equivalent reverse inequalities with the best possible constant factor  $K(\sigma)$ :

$$\sum_n \int_{\mathbf{R}_+^{i_0}} \csc h \left( \rho \|x\|_\alpha^{\eta} \|n - \tau\|_\beta^{\eta} \right) a_n f(x) dx > K(\sigma) \|f\|_{p,\tilde{\Phi}} \|a\|_{q,\Psi}, \quad (4.4)$$

$$\left( \sum_n \|n - \tau\|_{\beta}^{p\sigma - j_0} \left( \int_{\mathbf{R}_+^{i_0}} \csc h \left( \rho |x|_{\alpha}^{\eta} \|n - \tau\|_{\beta}^{\eta} \right) f(x) dx \right)^p \right)^{\frac{1}{p}} > K(\sigma) \|f\|_{p, \tilde{\Phi}}, \quad (4.5)$$

$$\begin{aligned} & \left( \int_{\mathbf{R}_+^{i_0}} \frac{|x|_{\alpha}^{q\sigma - i_0}}{(1 - \theta_{\sigma}(|x|_{\alpha}^{-1}))^{q-1}} \left( \sum_n \csc h \left( \rho |x|_{\alpha}^{\eta} \|n - \tau\|_{\beta}^{\eta} \right) a_n \right)^q dx \right)^{\frac{1}{q}} \\ & > K(\sigma) \|a\|_{q, \Psi}. \end{aligned} \quad (4.6)$$

In particular, for  $i_0 = j_0 = 1, \tau \in [0, \frac{1}{2}], \rho > 0, 0 < \eta < \sigma < 1$ , we set

$$\varphi(x) := x^{p(1-\sigma)-1}, \quad \tilde{\varphi}(x) = (1 - \theta_{\sigma}(x^{-1}))\varphi(x) (x \in \mathbf{R}_+), \quad \psi(n) = (n - \tau)^{q(1-\sigma)-1} (n \in \mathbf{N}).$$

(i) If  $p > 1$ , then we have the following equivalent inequalities with the best possible constant factor  $k(\sigma)$ :

$$\sum_{n=1}^{\infty} \int_0^{\infty} \csc h(\rho x^{\eta} (n - \tau)^{\eta}) a_n f(x) dx < k(\sigma) \|f\|_{p, \varphi} \|a\|_{q, \psi}, \quad (4.7)$$

$$\left( \sum_{n=1}^{\infty} (n - \tau)^{p\sigma-1} \left( \int_0^{\infty} \csc h(\rho x^{\eta} (n - \tau)^{\eta}) f(x) dx \right)^p \right)^{\frac{1}{p}} < k(\sigma) \|f\|_{p, \varphi}, \quad (4.8)$$

$$\left( \int_0^{\infty} x^{q\sigma-1} \left( \sum_{n=1}^{\infty} \csc h(\rho x^{\eta} (n - \tau)^{\eta}) a_n \right)^q dx \right)^{\frac{1}{q}} < k(\sigma) \|a\|_{q, \psi}; \quad (4.9)$$

(ii) if  $p < 0$  ( $0 < q < 1$ ), then we have the equivalent reverses of (4.7), (4.8) and (4.9) with the same best possible constant factor  $k(\sigma)$ ;

(iii) if  $0 < p < 1$  ( $q < 0$ ), then we have the following equivalent reverse inequalities with the best possible constant factor  $k(\sigma)$ :

$$\sum_{n=1}^{\infty} \int_0^{\infty} \csc h(\rho x^{\eta} (n - \tau)^{\eta}) a_n f(x) dx > k(\sigma) \|f\|_{p, \tilde{\varphi}} \|a\|_{q, \psi}, \quad (4.10)$$

$$\left( \sum_{n=1}^{\infty} (n - \tau)^{p\sigma-1} \left( \int_0^{\infty} \csc h(\rho x^{\eta} (n - \tau)^{\eta}) f(x) dx \right)^p \right)^{\frac{1}{p}} > k(\sigma) \|f\|_{p, \tilde{\varphi}}, \quad (4.11)$$

$$\left( \int_0^{\infty} \frac{x^{q\sigma-1}}{(1 - \theta_{\sigma}(x^{-1}))^{q-1}} \left( \sum_{n=1}^{\infty} \csc h(\rho x^{\eta} (n - \tau)^{\eta}) a_n \right)^q dx \right)^{\frac{1}{q}} > k(\sigma) \|a\|_{q, \psi}. \quad (4.12)$$

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