On a Functional Equation of Trigonometric Type

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Abstract

In this paper, we study the functional equation, \( f(x+y) - f(x)f(y) = d \sin x \sin y \).
Some generalizations of the above functional equation are also considered.

1 Introduction

In the fall of 1940, S. M. Ulam gave a wide-ranging talk before a Mathematical Colloquium at the University of Wisconsin in which he discussed a number of important unsolved problems. Among those was the following question concerning the stability of homomorphisms (cf. [15]):

Let \( G_1 \) be a group and let \( G_2 \) be a metric group with a metric \( d(\cdot, \cdot) \). Given \( \varepsilon > 0 \), does there exist a \( \delta > 0 \) such that if a function \( h : G_1 \to G_2 \) satisfies the inequality \( d(h(xy), h(x)h(y)) < \delta \) for all \( x, y \in G_1 \), then there is a homomorphism \( H : G_1 \to G_2 \) with \( d(h(x), H(x)) < \varepsilon \) for all \( x \in G_1 \)?

If the answer is affirmative, we say that the functional equation for homomorphisms is stable.

D. H. Hyers was the first mathematician to present the result concerning the stability of functional equations. He brilliantly answered the question of Ulam for the case where \( G_1 \) and \( G_2 \) are assumed to be Banach spaces (see [5]). This result of Hyers is stated as follows:

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Theorem 1.1 Let $f : E_1 \to E_2$ be a function between Banach spaces such that
\[ \|f(x + y) - f(x) - f(y)\| \leq \delta \] (1.1)
for some $\delta > 0$ and for all $x, y \in E_1$. Then the limit
\[ A(x) = \lim_{n \to \infty} 2^{-n} f(2^n x) \] (1.2)
exists for each $x \in E_1$, and $A : E_1 \to E_2$ is the unique additive function such that
\[ \|f(x) - A(x)\| \leq \delta \]
for every $x \in E_1$. Moreover, if $f(tx)$ is continuous in $t$ for each fixed $x \in E_1$, then the function $A$ is linear.

Taking this result into consideration, the additive Cauchy equation $f(x + y) = f(x) + f(y)$ is said to have the Hyers-Ulam stability on $(E_1; E_2)$ if for every $\delta > 0$ there exist a bounded subset $M$ of $E_2$ such that for every function $f : E_1 \to E_2$ satisfying inequality (1.1) there exists an additive function $A : E_1 \to E_2$ such that $f(x) - A(x) \in M$ for every $x \in E_1$, i.e., the difference $f - A$ is uniformly bounded.

For a broad study of the Hyers-Ulam stability for a large variety of functional equations the reader is referred to [3, 4, 6, 8, 10, 12].

In this paper, we will present some results concerning the solution as well as the Hyers-Ulam stability of the functional equation
\[ f(x + y) - f(x)f(y) = d \sin x \sin y, \] (1.3)
where $d$ is a real constant less than $-1$. Moreover, we introduce some functional equations of the form $f(x + y) + \lambda f(x)f(y) = \Phi(x, y)$ and then we investigate their stability properties (see [14]).

2 Preliminaries

In 2003, S. Butler [2] posed the following question:

Problem 2.1 Show that for $d < -1$ there are exactly two solutions $f : \mathbb{R} \to \mathbb{R}$ of the functional equation (1.3).

In 2004, M. Th. Rassias answered this question by proving the following theorem (see [11]):

Theorem 2.1 Let $d < -1$ be a constant. The functional equation (1.3) has exactly two solutions in the class of functions $f : \mathbb{R} \to \mathbb{R}$. More precisely, if a function $f : \mathbb{R} \to \mathbb{R}$ is a solution of Eq. (1.3), then $f$ has one of the forms
\[ f(x) = c \sin x + \cos x \quad \text{and} \quad f(x) = -c \sin x + \cos x, \]
where $c = \sqrt{-d - 1}$. 
Corollary 2.2 Let $d < -1$ be a constant. The functional equation
\begin{equation}
 g \left( x + y - \frac{\pi}{2} \right) - g(x) g(y) = d \cos x \cos y
\end{equation}
has exactly two solutions in the class of functions $g : \mathbb{R} \to \mathbb{R}$. More precisely, if a function $g : \mathbb{R} \to \mathbb{R}$ is a solution of Eq. (2.1), then $g$ has one of the forms
\[ g(x) = \sin x + c \cos x \quad \text{and} \quad f(x) = \sin x - c \cos x, \]
where $c = \sqrt{-d - 1}$.

Proof. Replacing in (2.1) $x, y$ by $\frac{\pi}{2} - x$ and $\frac{\pi}{2} - y$ respectively, we get
\[ g \left( \frac{\pi}{2} - x - y \right) - g \left( \frac{\pi}{2} - x \right) g \left( \frac{\pi}{2} - y \right) = d \sin x \sin y. \]
Now the function $f(x) = g \left( \frac{\pi}{2} - x \right)$ satisfies the functional equation (1.3). By Theorem 2.1,
\[ g \left( \frac{\pi}{2} - x \right) = \pm c \cos x + \sin x \]
and the conclusion follows by replacing back $x$ by $\frac{\pi}{2} - x$. \qed

Proof of Theorem 2.1. (M. Th. Rassias) Replacing $x$ with $x + z$ in (1.3), we get
\begin{equation}
 f(x + y + z) - f(x + z)f(y) - d \sin(x + z) \sin y = 0
\end{equation}
for all $x, y, z \in \mathbb{R}$. Similarly, if we replace $y$ with $y + z$ in (1.3), then we get
\begin{equation}
 f(x + y + z) - f(x)f(y + z) - d \sin x \sin(y + z) = 0
\end{equation}
for all $x, y, z \in \mathbb{R}$.

It follows from (2.2) and (2.3) that
\[ f(x) f(y + z) - f(x + z) f(y) + d \sin x \sin(y + z) - d \sin(x + z) \sin y = 0, \]
and hence
\begin{equation}
 f(x) \left( f(y + z) - f(y)f(z) - d \sin y \sin z \right) \\
 + f(x) f(y) f(z) + d f(x) \sin y \sin z \\
 - \left( f(x + z) - f(x)f(z) - d \sin x \sin z \right) f(y) \\
 - f(x)f(y)f(z) - d f(y) \sin x \sin z \\
 + d \sin x \sin(y + z) - d \sin(x + z) \sin y \\
 = f(x)f(y + z) - f(x + z)f(y) \\
 + d \sin x \sin(y + z) - d \sin(x + z) \sin y \\
 = 0
\end{equation}
for all $x, y, z \in \mathbb{R}$. \qed
Hence, it follows from (1.3) and (2.4) that

\[
\begin{align*}
    df(x) \sin y \sin z + d \sin x \sin(y + z) \\
    - df(y) \sin x \sin z - d \sin(x + z) \sin y \\
    = f(x)(f(y + z) - f(y)f(z) - d \sin y \sin z) \\
    + f(x)f(y)f(z) + df(x) \sin y \sin z \\
    - (f(x + z) - f(x)f(z) - d \sin x \sin z)f(y) \\
    - f(x)f(y)f(z) - df(y) \sin x \sin z \\
    + d \sin x \sin(y + z) - d \sin(x + z) \sin y \\
    + f(x)(- f(y + z) + f(y)f(z) + d \sin y \sin z) \\
    + f(y)(f(x + z) - f(x)f(z) - d \sin x \sin z)
\end{align*}
\]

\[(2.5)\]

for all \( x, y, z \in \mathbb{R} \).

If we set \( y = z = \pi/2 \) in the last equality, then

\[
f(x) - f\left(\frac{\pi}{2}\right) \sin x - \cos x = 0
\]

\[(2.6)\]

for any \( x \in \mathbb{R} \). Substituting \( \pi \) for \( x \) in (2.5) yields \( f(\pi) = -1 \). If we put \( x = y = \pi/2 \) in (1.3), then we obtain

\[
f\left(\frac{\pi}{2}\right)^2 = f(\pi) - d = -d - 1
\]

and hence

\[
f\left(\frac{\pi}{2}\right) = \sqrt{-d-1} = c \quad \text{or} \quad f\left(\frac{\pi}{2}\right) = -\sqrt{-d-1} = -c.
\]

Consequently, by (2.6), we have

\[
f(x) = c \sin x + \cos x \quad \text{or} \quad f(x) = -c \sin x + \cos x
\]

for all \( x \in \mathbb{R} \). \( \square \)

The Hyers-Ulam stability of the functional equation (1.3) was first proved in [7]:

**Theorem 2.3** Let \( d < -1 \) be a constant. There exists a constant \( K > 0 \) such that if \( 0 < \varepsilon < |d| \) and if a function \( f: \mathbb{R} \to \mathbb{R} \) satisfies the functional inequality

\[
|f(x + y) - f(x)f(y) - d \sin x \sin y| \leq \varepsilon
\]

(2.7)

for all \( x, y \in \mathbb{R} \), then

\[
|f(x) - f_0(x)| \leq K(\varepsilon + \sqrt{\varepsilon})
\]

(2.8)
holds for all $x \in \mathbb{R}$ and for some solution function $f_0$ of the functional equation (1.3). Here

$$K = (1 + \sqrt{2}) \frac{3 + \sqrt{1 + 8|d|} + |d|}{|d|}.$$ 

The original statement in [7] gives for $K$ the following formula

$$K = \max \left\{ \frac{3 + \sqrt{1 + 8|d|}}{|d|}, \ (1 + \sqrt{2}) \frac{3 + \sqrt{1 + 8|d|} + |d|}{|d|} \right\},$$

but we prove the following inequality

Lemma 2.4 The following inequality is valid for every real $d < -1$:

$$\left(1 + \sqrt{2}\right) \frac{3 + \sqrt{1 + 8|d|}}{|d|} \geq \frac{3 + \sqrt{1 + 8|d|}}{|d|}. \quad (2.9)$$

Proof. Indeed, the following inequality holds true for every real $t$ between $t_1 = -0.8701\cdots$ and $t_2 = 6.6985\cdots$

$$t < \left(1 + \sqrt{2}\right) \sqrt{1 + t},$$

since it is closely related to the quadratic equation $t^2 - (1 + \sqrt{2})^2 (1 + t) = 0$ ($t_1$ and $t_2$ are their roots). In particular, by taking

$$t = \frac{3 + \sqrt{1 + 8|d|}}{|d|} < 6,$$

inequality (2.9) follows.

We give now a stability result of the functional equation (2.1).

Theorem 2.5 Let $d < -1$ be a constant. There exists a constant $K > 0$ such that if $0 < \varepsilon < |d|$, and if a function $g : \mathbb{R} \to \mathbb{R}$ satisfies the functional inequality

$$\left| g \left( x + y - \frac{\pi}{2} \right) - g (x) g (y) - d \cos x \cos y \right| \leq \varepsilon,$$  
(2.10)

for all $x, y \in \mathbb{R}$, then there exists a choice of the sign $\pm$ below such that

$$|g (x) - (\sin x \pm c \cos x)| \leq K (\varepsilon + \sqrt{\varepsilon})$$  
(2.11)

holds for all $x \in \mathbb{R}$. Here

$$K = (1 + \sqrt{2}) \frac{3 + \sqrt{1 + 8|d|} + |d|}{|d|}.$$
Proof. By replacing in (2.10) \( x, y \) by \( \frac{\pi}{2} - x \) and \( \frac{\pi}{2} - y \) respectively, we obtain

\[
|g \left( \frac{\pi}{2} - x - y \right) - g \left( \frac{\pi}{2} - x \right) g \left( \frac{\pi}{2} - y \right) - d \sin x \sin y| \leq \varepsilon.
\]

As the function \( f(x) = g \left( \frac{\pi}{2} - x \right) \) satisfies

\[
|f(x + y) - f(x)y - d \sin x \sin y| \leq \varepsilon,
\]

Theorem 2.2 gives

\[
|f(x) - (\pm \varepsilon \sin x \cos x)| \leq K(\varepsilon + \sqrt{\varepsilon}).
\]

By replacing back \( x \) by \( \frac{\pi}{2} - x \), (2.11) follows. \( \square \)

We now introduce a theorem of J. Baker [1] concerning the superstability of the exponential functional equation.

**Theorem 2.6** Let \((G, +)\) be a semigroup and let \( \varepsilon > 0 \) be given. If a function \( f : G \to \mathbb{C} \) satisfies the inequality

\[
|f(x + y) - f(x)f(y)| \leq \varepsilon
\]

for all \( x, y \in G \), then either \( |f(x)| \leq (1 + \sqrt{1 + 4\varepsilon})/2 \) for all \( x \in G \) or \( f(x + y) = f(x)f(y) \) for all \( x, y \in G \).

In [9], using Theorem 2.6, it has been proved that each function \( f : \mathbb{R} \to \mathbb{R} \) satisfying the functional inequality (2.7) is bounded. Indeed, (2.8) yields

\[
|f(x)| \leq |f_0(x)| + K(\varepsilon + \sqrt{\varepsilon}) \leq 1 + \sqrt{|d| - 1} + K \left( |d| + \sqrt{|d|} \right).
\]

In the following lemma we present a better upper-bound for \( f \).

**Lemma 2.7** Let \( d \) and \( \varepsilon \) be real constants with \( d < -1 \) and \( 0 < \varepsilon < |d| \). If a function \( f : \mathbb{R} \to \mathbb{R} \) satisfies the functional inequality (2.7) for all \( x, y \in \mathbb{R} \), then

\[
|f(x)| \leq \frac{1 + \sqrt{1 + 8|d|}}{2}
\]

for all \( x \in \mathbb{R} \).

**Proof.** As \( 0 < \varepsilon < |d| \), it follows from (2.7) that

\[
-2|d| \leq f(x + y) - f(x)f(y) \leq 2|d|
\]

for all \( x, y \in \mathbb{R} \), which is equivalent to the inequality

\[
|f(x + y) - f(x)f(y)| \leq 2|d|
\]

for all \( x, y \in \mathbb{R} \).
According to Theorem 2.6, $f$ is either an exponential function or bounded. If $f$ were an exponential function, then it would follow from (2.7) that $|d \sin x \sin y| \leq \varepsilon$ for all $x, y \in \mathbb{R}$, which is contrary to our hypothesis, $\varepsilon < |d|$. Indeed, $f$ satisfies

$$|f(x)| \leq \frac{1 + \sqrt{1 + 8|d|}}{2}$$

for all $x \in \mathbb{R}$. 

By following the steps given in the proof of Theorem 2.1, we can easily prove the following lemma (see [7]).

**Lemma 2.8** Let $d$ and $\varepsilon$ be real constants with $d < -1$ and $0 < \varepsilon < |d|$. If a function $f : \mathbb{R} \rightarrow \mathbb{R}$ satisfies the functional inequality (2.7) for all $x, y \in \mathbb{R}$, then

$$\left| f(x) - f\left(\frac{\pi}{2}\right) \sin x - \cos x \right| \leq \frac{3 + \sqrt{1 + 8|d|}}{|d|} \varepsilon$$

for all $x \in \mathbb{R}$.

**Proof.** If we replace $x$ by $x + z$ in (2.7), then we have

$$|f(x + y + z) - f(x + z)f(y) - d \sin(x + z) \sin y| \leq \varepsilon$$  \hspace{1cm} (2.12)

for all $x, y, z \in \mathbb{R}$. Similarly, if we replace $y$ by $y + z$ in (2.7), then we get

$$|f(x + y + z) - f(x)f(y + z) - d \sin x \sin(y + z)| \leq \varepsilon$$  \hspace{1cm} (2.13)

for all $x, y, z \in \mathbb{R}$.

Using (2.12) and (2.13), we obtain

$$|f(x)f(y + z) - f(x + z)f(y) + d \sin x \sin(y + z)$$
$$- d \sin(x + z) \sin y|$$
$$= \left| [f(x + y + z) - f(x + z)f(y) - d \sin(x + z) \sin y]$$
$$- [f(x + y + z) - f(x)f(y + z) - d \sin x \sin(y + z)] \right|$$
$$\leq 2 \varepsilon$$  \hspace{1cm} (2.14)

for all $x, y, z \in \mathbb{R}$. It follows from (2.14) that

$$|f(x)[f(y + z) - f(y)f(z) - d \sin y \sin z]$$
$$+ f(x)f(y)f(z) + d f(x) \sin y \sin z$$
$$- [f(x + z) - f(x)f(z) - d \sin x \sin z]f(y)$$
$$- f(x)f(y)f(z) - d f(y) \sin x \sin z$$
$$+ d \sin x \sin(y + z) - d \sin(x + z) \sin y|$$
$$= \left| f(x)f(y + z) - f(x + z)f(y) + d \sin x \sin(y + z)$$
$$- d \sin(x + z) \sin y \right|$$
$$\leq 2 \varepsilon$$  \hspace{1cm} (2.15)
for all $x, y, z \in \mathbb{R}$.

It is easy to check that

$$
\begin{align*}
&|df(x)\sin y\sin z + d\sin x\sin(y + z) - df(y)\sin x\sin z - d\sin(x + z)\sin y| \\
= &\ |f(x)[f(y + z) - f(y)f(z) - d\sin y\sin z] + f(x)f(y)f(z) + d\sin x\sin y\sin z \\
&\ - [f(x + z) - f(x)f(z) - d\sin x\sin z]f(y) \\
&\ - f(x)f(y)f(z) - d\sin y\sin z\sin x \\
&\ + d\sin x\sin(y + z) - d\sin(x + z)\sin y \\
&\ - f(x)[f(y + z) - f(y)f(z) - d\sin y\sin z] \\
&\ + [f(x + z) - f(x)f(z) - d\sin x\sin z][f(y)].
\end{align*}
$$

Hence, in view of (2.15) and (2.7) we can now get

$$
\begin{align*}
&|df(x)\sin y\sin z + d\sin x\sin(y + z) - df(y)\sin x\sin z - d\sin(x + z)\sin y| \\
\leq &\ |f(x)[f(y + z) - f(y)f(z) - d\sin y\sin z] + f(x)f(y)f(z) + d\sin x\sin y\sin z \\
&\ - [f(x + z) - f(x)f(z) - d\sin x\sin z]f(y) \\
&\ - f(x)f(y)f(z) - d\sin y\sin z\sin x \\
&\ + d\sin x\sin(y + z) - d\sin(x + z)\sin y| \\
&\ + |f(x)||f(y + z) - f(y)f(z) - d\sin y\sin z| \\
&\ + |f(y)||f(x + z) - f(x)f(z) - d\sin x\sin z| \\
\leq &\ (2 + |f(x)| + |f(y)|)\varepsilon
\end{align*}
$$

for all $x, y, z \in \mathbb{R}$. If we set $y = z = \pi/2$ in the above inequality, then

$$
|df(x) - df\left(\frac{\pi}{2}\right)\sin x - d\cos x| \leq \left(2 + |f(x)| + \left|f\left(\frac{\pi}{2}\right)\right|\right)\varepsilon \tag{2.16}
$$

for all $x \in \mathbb{R}$.

If we assume that $f$ were unbounded, there should exist a sequence $\{x_n\} \subset \mathbb{R}$ such that $f(x_n) \neq 0$ for every $n \in \mathbb{N}$ and $|f(x_n)| \to \infty$ as $n \to \infty$. Set $x = x_n$ in (2.16), divide both sides of the resulting inequality by $|f(x_n)|$, and then let $n$ diverge to infinity. Then, we have $|d| \leq \varepsilon$ which is contrary to our hypothesis, say $\varepsilon < |d|$.

Therefore, $f$ must be bounded, and hence, $M_f := \sup_{x \in \mathbb{R}}|f(x)|$ has to be finite. Hence, it follows from (2.16) and Lemma 2.7 that

$$
|f(x) - f\left(\frac{\pi}{2}\right)\sin x - \cos x| \leq \frac{2(1 + M_f)}{|d|} \varepsilon \leq \frac{2 + (1 + \sqrt{1 + 8|d|})}{|d|} \varepsilon
$$

for all $x \in \mathbb{R}$. 

\qed
Lemma 2.9 Let $d$ and $\varepsilon$ be real constants with $d < -1$ and $0 < \varepsilon < |d|$. If a function $g : \mathbb{R} \to \mathbb{R}$ satisfies the functional inequality (2.10) for all $x, y \in \mathbb{R}$, then
\[ |g(x) - g(0) \cos x - \sin x| \leq \frac{3 + \sqrt{1 + 8|d|}}{|d|} \varepsilon \]  
for all $x \in \mathbb{R}$.

Proof. As we have already seen, if $g$ satisfies (2.10), then $f(x) = g\left(\frac{\pi}{2} - x\right)$ satisfies (2.7). By Lemma 2.7, we get
\[ |f(x) - f\left(\frac{\pi}{2}\right) \sin x - \cos x| \leq \frac{3 + \sqrt{1 + 8|d|}}{|d|} \varepsilon. \]
Replacing $x$ by $\frac{\pi}{2} - x$ in the above inequality, we get
\[ |f\left(\frac{\pi}{2} - x\right) - f\left(\frac{\pi}{2}\right) \sin \left(\frac{\pi}{2} - x\right) - \cos \left(\frac{\pi}{2} - x\right)| \leq \frac{3 + \sqrt{1 + 8|d|}}{|d|} \varepsilon, \]
which is (2.17).

3 Stability problems

In the following theorem, we prove the Hyers-Ulam stability of the functional equation (1.3), which is an improved version of Theorem 2.3. The dual result on the functional equation (2.1) is also presented.

Theorem 3.1 Let $d$ and $\varepsilon$ be real constants with $d < -1$ and $0 < \varepsilon < |d|$. If a function $f : \mathbb{R} \to \mathbb{R}$ satisfies the inequality (2.7) for all $x, y \in \mathbb{R}$, then there exists a solution function $f_0 : \mathbb{R} \to \mathbb{R}$ of the functional equation (1.3) such that
\[ |f(x) - f_0(x)| \leq \frac{(3 + \sqrt{1 + 8|d|})(1 + \sqrt{|d| - 1}) + |d| \varepsilon}{|d| |d| - 1} \]
for all $x \in \mathbb{R}$.

Proof. Let $0 < \varepsilon < |d|$ and let $f : \mathbb{R} \to \mathbb{R}$ satisfy the inequality (2.7) for all $x, y \in \mathbb{R}$. It follows from Lemma 2.8 that
\[ |f(x) - f\left(\frac{\pi}{2}\right) \sin x - \cos x| \leq \frac{3 + \sqrt{1 + 8|d|}}{|d|} \varepsilon \]  
for all $x \in \mathbb{R}$. Put $x = \pi$ in (3.1) to get
\[ |f(\pi) + 1| \leq \frac{3 + \sqrt{1 + 8|d|}}{|d|} \varepsilon. \]  
(3.2)
Furthermore, we set \( x = y = \pi/2 \) in (2.7) to obtain

\[
|f(\pi) - f\left(\frac{\pi}{2}\right)^2 - d| \leq \varepsilon. \quad (3.3)
\]

By combining (3.2) and (3.3), we get

\[
|f\left(\frac{\pi}{2}\right)^2 + d + 1| \leq \frac{3 + |d| + \sqrt{1 + 8|d|}}{|d|} \varepsilon.
\]

With \( c := \sqrt{|d| - 1} > 0 \), we have

\[
|f\left(\frac{\pi}{2}\right)^2 - c^2| \leq \frac{3 + |d| + \sqrt{1 + 8|d|}}{|d|} \varepsilon. \quad (3.4)
\]

Assume that \( f(\pi/2) \geq 0 \). It then follows from (3.4) that

\[
\left| f\left(\frac{\pi}{2}\right) - c \right| \leq \frac{3 + |d| + \sqrt{1 + 8|d|}}{|d||f(\pi/2) + c|} \varepsilon \leq \frac{3 + |d| + \sqrt{1 + 8|d|}}{c|d|} \varepsilon. \quad (3.5)
\]

Hence, (3.1) and (3.5) imply that

\[
|f(x) - c \sin x - \cos x| \\
\leq \left| f(x) - f\left(\frac{\pi}{2}\right) \sin x - \cos x \right| + \left| f\left(\frac{\pi}{2}\right) - c \right| \sin x \\
\leq \frac{(3 + \sqrt{1 + 8|d|})(1 + \sqrt{|d| - 1}) + |d|}{|d|\sqrt{|d| - 1}} \varepsilon
\]

for all \( x \in \mathbb{R} \).

Now, assume that \( f(\pi/2) < 0 \). It then follows from (3.4) that

\[
\left| f\left(\frac{\pi}{2}\right) + c \right| \leq \frac{3 + |d| + \sqrt{1 + 8|d|}}{|d||f(\pi/2) - c|} \varepsilon \leq \frac{3 + |d| + \sqrt{1 + 8|d|}}{c|d|} \varepsilon. \quad (3.6)
\]

So, we combine (3.1) and (3.6) to get

\[
|f(x) + c \sin x - \cos x| \\
\leq \left| f(x) - f\left(\frac{\pi}{2}\right) \sin x - \cos x \right| + \left| f\left(\frac{\pi}{2}\right) + c \right| \sin x \\
\leq \frac{(3 + \sqrt{1 + 8|d|})(1 + \sqrt{|d| - 1}) + |d|}{|d|\sqrt{|d| - 1}} \varepsilon
\]

for all \( x \in \mathbb{R} \).

We set

\[
f_0(x) := \begin{cases} 
    c \sin x + \cos x & \text{for } f(\pi/2) \geq 0, \\
    -c \sin x + \cos x & \text{for } f(\pi/2) < 0
\end{cases}
\]
for all $x \in \mathbb{R}$. In view of Theorem 2.1, both functions $c \sin x + \cos x$ and $-c \sin x + \cos x$ are solutions of the functional equation (1.3). Therefore, $f_0$ is a solution function of the functional equation (1.3).

The new upper bound given in Theorem 3.1,

$$\frac{(3 + \sqrt{1+8|d|})(1 + \sqrt{|d|-1}) + |d|}{|d|\sqrt{|d|-1}} \varepsilon,$$

is much smaller than that of Theorem 2.3 for sufficiently small $\varepsilon > 0$.

**Theorem 3.2** Let $d$ and $\varepsilon$ be real constants with $d < -1$ and $0 < \varepsilon < |d|$. If a function $g : \mathbb{R} \to \mathbb{R}$ satisfies the inequality (2.10) for all $x, y \in \mathbb{R}$, then there exists a solution function $g_0 : \mathbb{R} \to \mathbb{R}$ of the functional equation (2.1) such that

$$|g(x) - g_0(x)| \leq \frac{(3 + \sqrt{1+8|d|})(1 + \sqrt{|d|-1}) + |d|}{|d|\sqrt{|d|-1}} \varepsilon$$

for all $x \in \mathbb{R}$.

**Proof.** If $g$ satisfies (2.10), then $f(x) = g\left(\frac{x}{2} - x\right)$ satisfies the inequality (2.7). By Theorem 2.2, there exists a solution function $f_0 : \mathbb{R} \to \mathbb{R}$ of the functional equation (1.3) such that

$$|f(x) - f_0(x)| \leq \frac{(3 + \sqrt{1+8|d|})(1 + \sqrt{|d|-1}) + |d|}{|d|\sqrt{|d|-1}} \varepsilon,$$

for all $x \in \mathbb{R}$. By replacing back $x$ by $\frac{x}{2} - x$ in (3.7), we deduce

$$\left|g(x) - f_0\left(\frac{x}{2} - x\right)\right| \leq \frac{(3 + \sqrt{1+8|d|})(1 + \sqrt{|d|-1}) + |d|}{|d|\sqrt{|d|-1}} \varepsilon.$$

The proof is now completed taking into account that $g_0(x) = f_0\left(\frac{x}{2} - x\right)$ is a solution of the functional equation (2.1).

We will now prove a kind of generalized Hyers-Ulam stability of the functional equation (1.3).

**Theorem 3.3** Let $d$ be a real constant with $d < -1$ and let $M_1, M_2 : \mathbb{R} \to [-d, \infty)$ be functions. If a function $f : \mathbb{R} \to \mathbb{R}$ satisfies the inequality

$$|f(x+y) - f(x)f(y) - d \sin x \sin y| \leq \min\left\{M_1(x), M_2(y)\right\}$$

for all $x, y \in \mathbb{R}$, then $f$ is either an exponential function or bounded.
Proof. As $|d| \leq \min \{M_1(x), M_2(y)\}$ for all $x, y \in \mathbb{R}$, it follows from (3.8) that

$$|f(x + y) - f(x)f(y)| \leq \min \{2M_1(x), 2M_2(y)\}$$

for all $x, y \in \mathbb{R}$. In view of a theorem of L. Székelyhidi [13] (or [8, Corollary 9.4]), $f$ is either an exponential function or bounded. \qed

With $M_1(x) = M_2(y) = \varepsilon \geq |d|$ for all $x, y \in \mathbb{R}$, Theorem 3.3 reduces to the following corollary, which complements Theorem 3.1 for the case of $\varepsilon \geq |d|$.

**Corollary 3.4** Let $d$ and $\varepsilon$ be real constants with $d < -1$ and $\varepsilon \geq |d|$. If a function $f : \mathbb{R} \to \mathbb{R}$ satisfies the inequality (2.7) for all $x, y \in \mathbb{R}$, then $f$ is either an exponential function or bounded.

### 4 Generalization


$$f(x + y) + \lambda f(x)f(y) = \Phi(x, y),$$

(4.1)

where $f : \mathbb{R} \to \mathbb{C}$ is an unknown function, and they investigated their solutions. Here, we will introduce the solution of the functional equation (4.1) for the case $\Phi(x, y) = \psi(x)\psi(y)$.

**Theorem 4.1** Let $\lambda$ be a nonzero complex number and let $\psi : \mathbb{R} \to \mathbb{C}$ be a nonzero continuous function with $\psi(0) = 0$. If there exists a $c \in \mathbb{R}$ such that $\psi(c) \neq 0$ and

$$\psi(c)\psi(x + y) = \psi(x)\psi(y + c) + \psi(x + c)\psi(y)$$

for all $x, y \in \mathbb{R}$, then the functional equation

$$f(x + y) + \lambda f(x)f(y) = \psi(x)\psi(y),$$

for all $x, y \in \mathbb{R}$, has a solution of the form

$$f(x) = \pm \frac{1}{\lambda} \sqrt{\frac{\psi(3c)}{\psi(c)^3}} \psi(x) - \frac{1}{\lambda\psi(c)} \psi(x + c)$$

for all $x \in \mathbb{R}$.

Moreover, S.-E. Takahasi, T. Miura and H. Takagi [14] obtained the Hyers-Ulam stability results for the functional equation (4.1). We will now introduce a Hyers-Ulam stability theorem for the functional equation presented in Theorem 4.1.
Theorem 4.2 Let $\lambda$, $\psi$, and $c$ be as in Theorem 4.1, and suppose $\psi$ is bounded on $\mathbb{R}$. If a function $f : \mathbb{R} \to \mathbb{C}$ satisfies the inequality

$$|f(x + y) + \lambda f(x)f(y) - \psi(x)\psi(y)| \leq \epsilon$$

for all $x, y \in \mathbb{R}$ and for some $0 \leq \epsilon < |\psi(c)|^2$, then there exist a function $g : \mathbb{R} \to \mathbb{C}$ and a constant $K = K(\lambda, \psi, c)$ such that

$$g(x + y) + \lambda g(x)g(y) = \psi(x)\psi(y)$$

and

$$|f(x) - g(x)| \leq K(\epsilon + \sqrt{\epsilon})$$

for all $x, y \in \mathbb{R}$.

Takahasi et al. [14] considered the following two special cases of (4.1):

(I). $\Phi(x, y) = \phi(x + y)$,   
(II). $\Phi(x, y) = \psi(x)\psi(y)$   
\quad $\quad (x, y \in \mathbb{R})$.

In the next part of this paper, we discuss the following more general case of

$$\Phi(x, y) = \psi(x)\psi(y) + \phi(x + y),$$

that is the equation

$$f(x + y) + \lambda f(x)f(y) = \psi(x)\psi(y) + \phi(x + y).$$

(4.2)

Theorem 4.3 Let $f$, $\psi$, $\phi : \mathbb{R} \to \mathbb{C}$ with $\psi(0) \neq 0$, satisfying (4.2) for all $x, y \in \mathbb{R}$. Assume that $\lambda \in \mathbb{R}$ is such that

$$\lambda \left[ f(0) + \phi(0) + \psi^2(0) \right] \neq -1.$$  

(4.3)

Then there exists $\beta \in \mathbb{C}$ such that

$$\phi(x) = \beta\psi(x).$$  

(4.4)

Proof. By taking $y = 0$ in (4.2), we get

$$f(x) + \lambda f(x)f(0) = \psi(x)\psi(0) + \phi(x),$$

or

$$(1 + \lambda f(0)) f(x) = \phi(x) + \psi(x)\psi(0)$$  

(4.5)

If $1 + \lambda f(0) = 0$, then $\phi(x) + \psi(x)\psi(0) = 0$ and the conclusion follows with $\beta = -\psi(0)$.

Let now $1 + \lambda f(0) \neq 0$. By (4.5),

$$f(x) = a\phi(x) + b\psi(x),$$
where
\[ a = \frac{1}{1 + \lambda f(0)}, \quad b = \frac{\psi(0)}{1 + \lambda f(0)} \] (4.6)

By replacing in (4.2), we get
\[ a\phi(x + y) + b\psi(x + y) + \lambda [a\phi(x) + b\psi(x)] \left[ a\phi(y) + b\psi(y) \right] = \psi(x)\psi(y) + \phi(x + y). \]

With \( y = 0 \), we obtain
\[ a\phi(x) + b\psi(x) + \lambda [a\phi(x) + b\psi(x)] \left[ a\phi(0) + b\psi(0) \right] = \psi(x)\psi(0) + \phi(x), \]
which is
\[ [a + \lambda a(a\phi(0) + b\psi(0)) - 1] \phi(x) = [\psi(0) - b - \lambda b(a\phi(0) + b\psi(0))] \psi(x). \] (4.7)

According to (4.3) the coefficient of \( \phi(x) \) is non-zero, since
\[ a + \lambda a(a\phi(0) + b\psi(0)) - 1 = \frac{1}{[1 + \lambda f(0)]^2} \left\{ 1 + \lambda \left[ f(0) + \phi(0) + \psi^2(0) \right] \right\} \]
\[ \neq 0. \]

By (4.7), we get
\[ \phi(x) = \frac{\psi(0) - b - \lambda b(a\phi(0) + b\psi(0))}{a + \lambda a(a\phi(0) + b\psi(0)) - 1} \psi(x) \]
and the conclusion follows. \( \square \)

According to Theorem 4.3, (4.4) is a necessary condition for (4.2) to have non-trivial solutions. It is now natural to discuss the following equation
\[ f(x + y) + \lambda f(x)f(y) = \psi(x)\psi(y) + \beta \psi(x + y), \] (4.8)
where \( \beta \in \mathbb{R} \) is non-zero (\( \beta = 0 \) case is treated in [14]).

**Theorem 4.4** Let there be given \( \lambda, \beta \in \mathbb{C} \) such that \( \lambda \beta^2 \neq 1 \) and a function \( \psi : \mathbb{R} \to \mathbb{C} \) different from identical zero function. Then the functional equation (4.8) admits solutions \( f : \mathbb{R} \to \mathbb{C} \) different from identical zero function if and only if \( \psi(x) = me^{\alpha x} \), for some \( m, \alpha \in \mathbb{C}, m \neq 0 \). In this case, the functional equation (4.8) admits the solutions
\[ f(x) = pe^{\alpha x}, \] (4.9)
where
\[ p = \frac{1}{2\lambda} \left( -1 \pm \sqrt{4m^2\lambda + 4m\beta \lambda + 1} \right) \]
Proof. Taking $y = 0$ in (4.8), we get
\[ f(x) + \lambda f(x) f(0) = \psi(x) \psi(0) + \beta \psi(x), \]
or
\[ [1 + \lambda f(0)] f(x) = [\beta + \psi(0)] \psi(x). \]
The coefficients of $f$ and $\psi$ are non-zero, otherwise $f \equiv 0$, or $\psi \equiv 0$, which is unacceptable.

Hence
\[ f(x) = k \psi(x), \quad (4.10) \]
where
\[ k = \frac{\beta + \psi(0)}{1 + \lambda f(0)}. \]

By replacing in (4.8), we deduce that
\[ k \psi(x + y) + \lambda k^2 \psi(x) \psi(y) = \psi(x) \psi(y) + \beta \psi(x + y), \]
or
\[ (k - \beta) \psi(x + y) + (\lambda k^2 - 1) \psi(x) \psi(y) = 0 \quad (4.11) \]
If $k - \beta = 0$, then $\lambda k^2 - 1 = 0$, otherwise it results to $\psi \equiv 0$. Thus $\lambda \beta^2 = 1$, unacceptable.

In consequence, $1 - \beta k \neq 0$. By (4.11),
\[ \psi(x + y) - \frac{1}{m} \psi(x) \psi(y) = 0, \]
where $m = \frac{\beta - k}{\lambda k^2 - 1}$. Using [14, Lemma 1], we deduce that there exists $\alpha \in \mathbb{C}$ such that
\[ \psi(x) = m e^{\alpha x}. \]
Furthermore, (4.10) gives $f(x) = k m e^{\alpha x}$. By replacing these $f$ and $\psi$ in (4.8), we get
\[ (\lambda m k^2 + k - m - \beta) e^{\alpha(x+y)} = 0. \]

Now (4.9) follows and the proof is completed. \( \square \)

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