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UNIQUENESS THEOREMS ON FUNCTIONAL INEQUALITIES CONCERNING CUBIC–QUADRATIC–ADDITIVE EQUATION

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Abstract. We prove uniqueness theorems concerning the functional inequalities in connection with an n-dimensional cubic-quadratic-additive equation $\sum_{i=1}^{m} c_i f\left(a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n\right) = 0$ by applying the direct method.

1. Introduction

Let V and W be real vector spaces. For a given mapping $f : V \to W$, we define

$$\begin{aligned}
Af(x,y) &:= f(x+y) - f(x) - f(y),\\
Qf(x,y) &:= f(x+y) + f(x-y) - 2f(x) - 2f(y),\\
Cf(x,y) &:= f(x+2y) - 3f(x+y) - f(x-y) + 3f(x) - 6f(y)
\end{aligned}$$

for all $x,y \in V$. A mapping $f : V \to W$ is called an additive mapping, a quadratic mapping, or a cubic mapping provided f satisfies the functional equation $Af(x,y) = 0$ for all $x,y \in V$, $Qf(x,y) = 0$ for all $x,y \in V$, or $Cf(x,y) = 0$ for all $x,y \in V$, respectively. We note that the mappings $g,h,k : \mathbb{R} \to \mathbb{R}$ given by $g(x) = ax$, $h(x) = ax^2$, and $k(x) = ax^3$ are solutions of $Ag(x,y) = 0$, $Qh(x,y) = 0$, and $Ck(x,y) = 0$, respectively.

A mapping $f : V \to W$ is called a cubic-quadratic-additive mapping if and only if f is represented by the sum of an additive mapping, a quadratic mapping, and a cubic mapping. A functional equation is called a cubic-quadratic-additive functional equation provided that each of its solutions is a cubic-quadratic-additive mapping and every cubic-quadratic-additive mapping is also a solution of that equation. The mapping $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = ax^3 + bx^2 + cx$ is a solution of the cubic-quadratic-additive functional equation.

For the study of functional inequalities concerning the cubic-quadratic-additive equations and a broad variety of other types of functional inequalities, the reader is referred to [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 15, 16, 17, 18].

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Throughout this paper, let V and W be real vector spaces, X a real normed space, Y a real Banach space, and let $\mathbb{N}_0$ denote the set of all nonnegative integers. For a given mapping $f : V \to W$, we define $Df : V^n \to W$ by

$$Df(x_1,x_2,\ldots,x_n) := \sum_{i=1}^{m} c_i f\big(a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n\big) \tag{1.1}$$

for all $x_1,x_2,\ldots,x_n \in V$, where m is a positive integer and c_i, a_{ij} are real constants.

In this paper, we prove uniqueness theorems that can be easily applied to the investigation of functional inequalities concerning a large class of functional equations of the form $Df(x_1,x_2,\ldots,x_n) = 0$, which includes the cubic-quadratic-additive functional equation as a special case. This theorem is particularly useful for proving the Hyers-Ulam stability of a variety of functional equations.

2. Preliminaries

For a given mapping $f : V \to W$, we use the following abbreviations:

$$f_o(x) := \frac{f(x) - f(-x)}{2}, \qquad f_e(x) := \frac{f(x) + f(-x)}{2},$$

$$f_o^{(1)}(x) := \frac{a^3 f_o(x) - f_o(ax)}{a^3 - a}, \quad f_o^{(2)}(x) := -\frac{a f_o(x) - f_o(ax)}{a^3 - a}$$

for all $x \in V$. We will now introduce a lemma that was proved in [14, Corollary 2].

LEMMA 2.1. *Let $k > 1$ be a real constant, let $\phi : V \setminus \{0\} \to [0,\infty)$ be a function satisfying either*

$$\Phi(x) := \sum_{i=0}^{\infty} \frac{1}{k^i} \phi(k^i x) < \infty \tag{2.1}$$

for all $x \in V \setminus \{0\}$ or

$$\Phi(x) := \sum_{i=0}^{\infty} k^{3i} \phi\left(\frac{x}{k^i}\right) < \infty \tag{2.2}$$

for all $x \in V \setminus \{0\}$, and let $f : V \to Y$ be an arbitrarily given mapping. If there exists a mapping $F : V \to Y$ satisfying

$$\|f(x) - F(x)\| \leqslant \Phi(x) \tag{2.3}$$

for all $x \in V \setminus \{0\}$ and

$$F_o^{(1)}(kx) = kF_o^{(1)}(x), \quad F_e(kx) = k^2 F_e(x), \quad F_o^{(2)}(kx) = k^3 F_o^{(2)}(x) \tag{2.4}$$

for all $x \in V$, then F is a unique mapping satisfying (2.3) and (2.4).

We introduce lemmas that were proved in [14, Corollary 3].

LEMMA 2.2. *Let $k>1$ be a real number, let $\phi,\psi : V\setminus\{0\}\to[0,\infty)$ be functions satisfying each of the following conditions*

$$\sum_{i=0}^{\infty} k^i \psi\left(\frac{x}{k^i}\right) < \infty, \qquad \sum_{i=0}^{\infty} \frac{1}{k^{2i}} \phi\left(k^i x\right) < \infty,$$
$$\tilde{\Phi}(x) := \sum_{i=0}^{\infty} k^i \phi\left(\frac{x}{k^i}\right) < \infty, \quad \tilde{\Psi}(x) := \sum_{i=0}^{\infty} \frac{1}{k^{2i}} \psi\left(k^i x\right) < \infty$$

for all $x\in V\setminus\{0\}$, and let $f : V\to Y$ be an arbitrarily given mapping. If there exists a mapping $F : V\to Y$ satisfying the inequality

$$\|f(x)-F(x)\| \leqslant \tilde{\Phi}(x)+\tilde{\Psi}(x) \tag{2.5}$$

for all $x\in V\setminus\{0\}$ and the conditions in (2.4) for all $x\in V$, then F is a unique mapping satisfying the conditions (2.4) for all $x\in V$ and the inequality (2.5) for all $x\in V\setminus\{0\}$.

LEMMA 2.3. *Let $k>1$ be a real number, let $\phi,\psi : V\setminus\{0\}\to[0,\infty)$ be functions satisfying each of the following conditions*

$$\sum_{i=0}^{\infty} k^{2i} \psi\left(\frac{x}{k^i}\right) < \infty, \qquad \sum_{i=0}^{\infty} \frac{1}{k^{3i}} \phi\left(k^i x\right) < \infty,$$
$$\tilde{\Phi}(x) := \sum_{i=0}^{\infty} k^{2i} \phi\left(\frac{x}{k^i}\right) < \infty, \quad \tilde{\Psi}(x) := \sum_{i=0}^{\infty} \frac{1}{k^{3i}} \psi\left(k^i x\right) < \infty$$

for all $x\in V\setminus\{0\}$, and let $f : V\to Y$ be an arbitrarily given mapping. If there exists a mapping $F : V\to Y$ satisfying the inequality

$$\|f(x)-F(x)\| \leqslant \tilde{\Phi}(x)+\tilde{\Psi}(x) \tag{2.6}$$

for all $x\in V\setminus\{0\}$ and the conditions in (2.4) for all $x\in V$, then F is a unique mapping satisfying the conditions (2.4) for all $x\in V$ and the inequality (2.6) for all $x\in V\setminus\{0\}$.

3. Main results

In the following four theorems, we prove that there exists only one exact solution near every approximate solution to $Df(x_1,x_2,\ldots,x_n)=0$.

THEOREM 3.1. *Let a be a real constant with $a\notin\{-1,0,1\}$, let n be a fixed integer greater than 1, let $\mu,\nu : V\setminus\{0\}\to[0,\infty)$ be functions satisfying the conditions*

$$\begin{cases} \displaystyle\sum_{i=0}^{\infty} \frac{\mu(a^i x)}{|a|^{3i}} < \infty \quad \text{and} \quad \sum_{i=0}^{\infty} \frac{\nu(a^i x)}{|a|^{3i}} < \infty \text{ when } |a|<1, \\ \displaystyle\sum_{i=0}^{\infty} \frac{\mu(a^i x)}{|a|^{i}} < \infty \quad \text{and} \quad \sum_{i=0}^{\infty} \frac{\nu(a^i x)}{|a|^{i}} < \infty \text{ when } |a|>1 \end{cases} \tag{3.1}$$

for all $x \in V \setminus \{0\}$, *and let* $\varphi : (V \setminus \{0\})^n \to [0,\infty)$ *be a function satisfying the condition*

$$\begin{cases} \sum_{i=0}^{\infty} \frac{\varphi(a^i x_1, a^i x_2, \ldots, a^i x_n)}{|a|^{3i}} < \infty \text{ when } |a| < 1, \\ \sum_{i=0}^{\infty} \frac{\varphi(a^i x_1, a^i x_2, \ldots, a^i x_n)}{|a|^{i}} < \infty \text{ when } |a| > 1 \end{cases} \tag{3.2}$$

for all $x_1, x_2, \ldots, x_n \in V \setminus \{0\}$. *If a mapping* $f : V \to Y$ *satisfies* $f(0) = 0$,

$$\|f_e(ax) - a^2 f_e(x)\| \leqslant \mu(x) \quad \text{and} \quad \|f_o(a^2x) - (a+a^3) f_o(ax) + a^4 f_o(x)\| \leqslant \nu(x) \tag{3.3}$$

for all $x \in V \setminus \{0\}$, *and if* f *moreover satisfies*

$$\|Df(x_1, x_2, \ldots, x_n)\| \leqslant \varphi(x_1, x_2, \ldots, x_n) \tag{3.4}$$

for all $x_1, x_2, \ldots, x_n \in V \setminus \{0\}$, *then there exists a unique mapping* $F : V \to Y$ *such that*

$$DF(x_1, x_2, \ldots, x_n) = 0 \tag{3.5}$$

for all $x_1, x_2, \ldots, x_n \in V \setminus \{0\}$, *and*

$$F_e(ax) = a^2 F_e(x), \quad F_o^{(1)}(ax) = aF_o^{(1)}(x), \quad F_o^{(2)}(ax) = a^3 F_o^{(2)}(x) \tag{3.6}$$

for all $x \in V$, *and such that*

$$\|f(x) - F(x)\| \leqslant \sum_{i=0}^{\infty} \left(\frac{\mu(a^i x)}{a^{2i+2}} + \frac{|a^{2i+2} - 1| \nu(a^i x)}{|a^3 - a||a|^{3i+3}} \right) \tag{3.7}$$

for all $x \in V \setminus \{0\}$.

Proof. First, we define $A := \{f : V \to Y \mid f(0) = 0\}$ and a mapping $J_m : A \to A$ by

$$J_m f(x) := \frac{f_o^{(2)}(a^m x)}{a^{3m}} + \frac{f_e(a^m x)}{a^{2m}} + \frac{f_o^{(1)}(a^m x)}{a^m}$$

for $x \in V$ and $m \in \mathbb{N}_0$. It follows from (3.3) that

$$\begin{aligned} &\|J_m f(x) - J_{m+l} f(x)\| \\ &\leqslant \sum_{i=m}^{m+l-1} \|J_i f(x) - J_{i+1} f(x)\| \\ &= \sum_{i=m}^{m+l-1} \left\| \frac{f_o(a^{i+1}x) - af_o(a^i x)}{(a^3 - a)a^{3i}} - \frac{f_o(a^{i+2}x) - af_o(a^{i+1}x)}{(a^3 - a)a^{3i+3}} \right. \\ &\qquad + \frac{f_e(a^i x)}{a^{2i}} - \frac{f_e(a^{i+1}x)}{a^{2i+2}} - \frac{f_o(a^{i+1}x) - a^3 f_o(a^i x)}{(a^3 - a)a^i} \\ &\qquad \left. + \frac{f_o(a^{i+2}x) - a^3 f_o(a^{i+1}x)}{(a^3 - a)a^{i+1}} \right\| \end{aligned} \tag{3.8}$$

$$
\begin{aligned}
&\leqslant \sum_{i=m}^{m+l-1}\left\|\frac{f_e(a^{i+1}x)-a^2 f_e(a^i x)}{a^{2i+2}}\right\| \\
&\quad+\sum_{i=m}^{m+l-1}\left\|-\frac{f_o(a^2a^ix)-(a+a^3)f_o(a^{i+1}x)+a^4 f_o(a^i x)}{(a^3-a)a^{3i+3}}\right. \\
&\qquad\qquad\left.+\frac{f_o(a^2a^ix)-(a+a^3)f_o(a^{i+1}x)+a^4 f_o(a^i x)}{(a^3-a)a^{i+1}}\right\| \\
&\leqslant \sum_{i=m}^{m+l-1}\left(\frac{\mu(a^i x)}{a^{2i+2}}+\frac{|a^{2i+2}-1|\nu(a^i x)}{|a^3-a||a|^{3i+3}}\right)
\end{aligned}
$$

for all $x \in V\setminus\{0\}$.

In view of (3.1) and (3.8), the sequence $\{J_m f(x)\}$ is a Cauchy sequence for all $x \in V\setminus\{0\}$. Since Y is complete and $f(0)=0$, the sequence $\{J_m f(x)\}$ converges for all $x\in V$. Hence, we can define a mapping $F: V \to Y$ by

$$
F(x) := \lim_{m\to\infty} J_m f(x) = \lim_{m\to\infty}\left(\frac{f_o^{(2)}(a^m x)}{a^{3m}}+\frac{f_e(a^m x)}{a^{2m}}+\frac{f_o^{(1)}(a^m x)}{a^m}\right)
$$

for all $x\in V$.

We easily obtain from the definition of F that

$$
\begin{aligned}
F_o^{(1)}(ax) &= -\frac{F_o(a^2x)-a^3F_o(ax)}{a^3-a} \\
&= -\lim_{m\to\infty}\left(\frac{f_o(a^{m+3}x)-af_o(a^{m+2}x)}{a^{3m}(a^3-a)^2}-\frac{f_o(a^{m+3}x)-a^3f_o(a^{m+2}x)}{a^m(a^3-a)^2}\right) \\
&\quad+\lim_{m+1\to\infty} a^3\left(\frac{f_o(a^{(m+1)+2}x)-af_o(a^{(m+1)+1}x)}{a^{3(m+1)}(a^3-a)^2}\right. \\
&\qquad\left.-\frac{f_o(a^{(m+1)+2}x)-a^3f_o(a^{(m+1)+1}x)}{a^{(m+1)}(a^3-a)^2}\right) \\
&= -\lim_{m\to\infty}\left(\frac{f_o(a^{m+3}x)-a^3f_o(a^{m+2}x)}{a^{m+1}(a^3-a)}\right) \\
&= -\lim_{m\to\infty}\left(\frac{f_o(a^{m+2}ax)-a^3f_o(a^{m+1}ax)}{a^{m+1}(a^3-a)}\right) \\
&= -\lim_{m+1\to\infty} a\left(\frac{f_o(a^{(m+1)+2}x)-a^3f_o(a^{(m+1)+1}x)}{a^{(m+1)+1}(a^3-a)}\right) \\
&= aF_o^{(1)}(x),
\end{aligned}
$$

$$
\begin{aligned}
F_e(ax) &= \frac{F(ax)+F(-ax)}{2} \\
&= \lim_{m\to\infty}\frac{f(a^{m+1}x)+f(-a^{m+1}x)}{2a^{2m}}
\end{aligned}
$$

$$= a^2 \lim_{m+1\to\infty} \frac{f(a^{m+1}x) + f(-a^{m+1}x)}{2a^{2(m+1)}}$$
$$= a^2 F_e(x),$$

$$\begin{aligned}
F_o^{(2)}(ax) &= \frac{F(a^2x) - aF(ax)}{a^3 - a} \\
&= \lim_{m\to\infty} \left(\frac{f_o(a^{m+3}x) - af_o(a^{m+2}x)}{a^{3m}(a^3-a)^2} - \frac{f_o(a^{m+3}x) - a^3 f_o(a^{m+2}x)}{a^m(a^3-a)^2} \right) \\
&\quad - \lim_{m+1\to\infty} a \left(\frac{f_o(a^{(m+1)+2}x) - af_o(a^{(m+1)+1}x)}{a^{3(m+1)}(a^3-a)^2} \right. \\
&\qquad \left. - \frac{f_o(a^{(m+1)+2}x) - a^3 f_o(a^{(m+1)+1}x)}{a^{(m+1)}(a^3-a)^2} \right) \\
&= \lim_{m\to\infty} \left(\frac{f_o(a^{m+2}ax) - af_o(a^{m+1}ax)}{a^{3m+3}(a^3-a)} \right) \\
&= a^3 \lim_{m+1\to\infty} \left(\frac{f_o(a^{(m+1)+2}x) - af_o(a^{(m+1)+1}x)}{a^{3(m+1)+3}(a^3-a)} \right) \\
&= a^3 F_o^{(2)}(x)
\end{aligned}$$

for all $x \in V$, and by (1.1) and (3.2), we get

$$\begin{aligned}
&\|DF(x_1, x_2, \ldots, x_n)\| \\
&= \lim_{m\to\infty} \left\| \frac{Df_o\left(a^{m+1}x_1, \ldots, a^{m+1}x_n\right) - aDf_o\left(a^m x_1, \ldots, a^m x_n\right)}{a^{3m}(a^3-a)} \right. \\
&\qquad + \frac{Df_e\left(a^m x_1, \ldots, a^m x_n\right)}{a^{2m}} \\
&\qquad \left. - \frac{Df_o\left(a^{m+1}x_1, \ldots, a^{m+1}x_n\right) - a^3 Df_o\left(a^m x_1, \ldots, a^m x_n\right)}{a^m(a^3-a)} \right\| \\
&\leqslant \lim_{m\to\infty} \left(\left(\frac{1}{|a|^{2m}} + \frac{|a|^{2m+3} + a}{|a^3-a||a|^{3m}} \right) \varphi_e\left(a^m x_1, \ldots, a^m x_n\right) \right. \\
&\qquad \left. + \frac{|a|^{2m} + 1}{|a^3-a||a|^{3m}} \varphi_e\left(a^{m+1}x_1, \ldots, a^{m+1}x_n\right) \right) \\
&= 0
\end{aligned}$$

for all $x_1, x_2, \ldots, x_n \in V \setminus \{0\}$, where $\varphi_e(x_1, \ldots, x_n) := \frac{\varphi(x_1,\ldots,x_n) + \varphi(-x_1,\ldots,-x_n)}{2}$; *i.e.*,

$$DF(x_1, x_2, \ldots, x_n) = 0$$

for all $x_1, x_2, \ldots, x_n \in V \setminus \{0\}$. Moreover, if we put $m = 0$ and let $l \to \infty$ in (3.8), then we obtain the inequality (3.7).

Notice that the equalities

$$F_o^{(1)}(|a|x) = |a|F_o^{(1)}(x), \quad F_o^{(1)}\left(\frac{x}{|a|}\right) = \frac{F_o^{(1)}(x)}{|a|},$$
$$F_e(|a|x) = |a|^2 F_e(x), \quad F_e\left(\frac{x}{|a|}\right) = \frac{F_e(x)}{|a|^2}, \tag{3.9}$$
$$F_o^{(2)}(|a|x) = |a|^3 F_o^{(2)}(x), \quad F_o^{(2)}\left(\frac{x}{|a|}\right) = \frac{F_o^{(2)}(x)}{|a|^3}$$

are true in view of (3.6). Therefore, the equalities in (2.4) hold, for all $x \in V$, with $k = |a|$ if $|a| > 1$ or $k = \frac{1}{|a|}$ if $|a| < 1$.

When $|a| > 1$, in view of Lemma 2.1, there exists a unique mapping $F : V \to Y$ satisfying the equalities in (3.6) and the inequality (3.7), since the inequality

$$\begin{aligned}\|f(x) - F(x)\| &\leqslant \sum_{i=0}^{\infty}\left(\frac{\mu(a^i x)}{a^{2i+2}} + \frac{|a^{2i+2}-1|\nu(a^i x)}{|a^3-a||a|^{3i+3}}\right)\\ &\leqslant \sum_{i=0}^{\infty}\left(\frac{\mu(a^i x)}{|a|^i} + \frac{\nu(a^i x)}{|a^3-a||a|^i}\right)\\ &\leqslant \sum_{i=0}^{\infty}\frac{\phi(k^i x)}{k^i}\end{aligned}$$

holds for all $x \in V\setminus\{0\}$, where we set $k := |a|$ and $\phi(x) := \mu(x) + \mu(-x) + \frac{\nu(x)+\nu(-x)}{|a^3-a|}$.

When $|a| < 1$, in view of Lemma 2.1, there exists a unique mapping $F : V \to Y$ satisfying the equalities in (3.6) and the inequality (3.7), since the inequality

$$\begin{aligned}\|f(x) - F(x)\| &\leqslant \sum_{i=0}^{\infty}\left(\frac{\mu(a^i x)}{a^{2i+2}} + \frac{|a^{2i+2}-1|\nu(a^i x)}{|a^3-a||a|^{3i+3}}\right)\\ &\leqslant \sum_{i=0}^{\infty}\left(\frac{\mu(a^i x)}{|a|^{3i+3}} + \frac{\nu(a^i x)}{|a^3-a||a|^{3i+3}}\right)\\ &\leqslant \sum_{i=0}^{\infty} k^{3i}\phi\left(\frac{x}{k^i}\right)\end{aligned}$$

holds for all $x \in V\setminus\{0\}$, where $k := \frac{1}{|a|}$ and $\phi(x) := \frac{\mu(x)+\mu(-x)}{|a|^3} + \frac{\nu(x)+\nu(-x)}{|a^3-a||a|^3}$. □

In the following theorem, we assume that μ, ν and φ satisfy other conditions than those of Theorem 3.1 and we prove that there exists a unique exact solution near every approximate solution to $Df(x_1, x_2, \ldots, x_n) = 0$.

THEOREM 3.2. *Assume that a is a real constant with $a \notin \{-1, 0, 1\}$. Let n be a fixed integer greater than 1, let $\mu, \nu : V\setminus\{0\} \to [0,\infty)$ be functions satisfying the* *conditions*

$$\begin{cases} \sum_{i=0}^{\infty} |a|^i \mu\left(\frac{x}{a^i}\right) < \infty \quad \textit{and} \quad \sum_{i=0}^{\infty} |a|^i \nu\left(\frac{x}{a^i}\right) < \infty \quad \textit{when } |a| < 1, \\ \sum_{i=0}^{\infty} |a|^{3i} \mu\left(\frac{x}{a^i}\right) < \infty \quad \textit{and} \quad \sum_{i=0}^{\infty} |a|^{3i} \nu\left(\frac{x}{a^i}\right) < \infty \textit{ when } |a| > 1 \end{cases} \tag{3.10}$$

for all $x \in V \setminus \{0\}$, *and let* $\varphi : (V \setminus \{0\})^n \to [0, \infty)$ *be a function satisfying the condition*

$$\begin{cases} \sum_{i=0}^{\infty} |a|^i \varphi\left(\frac{x_1}{a^i}, \frac{x_2}{a^i}, \ldots, \frac{x_n}{a^i}\right) < \infty \quad \textit{when } |a| < 1, \\ \sum_{i=0}^{\infty} |a|^{3i} \varphi\left(\frac{x_1}{a^i}, \frac{x_2}{a^i}, \ldots, \frac{x_n}{a^i}\right) < \infty \textit{ when } |a| > 1 \end{cases} \tag{3.11}$$

for all $x_1, x_2, \ldots, x_n \in V \setminus \{0\}$. *If a mapping* $f : V \to Y$ *satisfies* $f(0) = 0$, *the inequalities in* (3.3) *for all* $x \in V \setminus \{0\}$, *and* (3.4) *for all* $x_1, x_2, \ldots, x_n \in V \setminus \{0\}$, *then there exists a unique mapping* $F : V \to Y$ *satisfying* (3.5) *for all* $x_1, x_2, \ldots, x_n \in V \setminus \{0\}$ *and* (3.6) *for all* $x \in V$, *and such that*

$$\|f(x) - F(x)\| \leqslant \sum_{i=0}^{\infty} \left(a^{2i} \mu\left(\frac{x}{a^{i+1}}\right) + \frac{|a^{3i+3} - a^{i+1}|}{|a^3 - a|} \nu\left(\frac{x}{a^{i+2}}\right) \right) \tag{3.12}$$

for all $x \in V \setminus \{0\}$.

Proof. First, we define the mappings $J_m f : V \to Y$ by

$$J_m f(x) := a^{3m+3} f_o^{(2)}\left(\frac{x}{a^{m+1}}\right) + a^{2m} f_e\left(\frac{x}{a^m}\right) + a^{m+1} f_o^{(1)}\left(\frac{x}{a^{m+1}}\right)$$

for all $x \in V$ and $m \in \mathbb{N}_0$. It follows from (3.3) that

$$\begin{aligned} &\|J_m f(x) - J_{m+l} f(x)\| \\ &\leqslant \sum_{i=m}^{m+l-1} a^{2i} \left\| f_e\left(a \frac{x}{a^{i+1}}\right) - a^2 f_e\left(\frac{x}{a^{i+1}}\right) \right\| \\ &\quad + \frac{1}{|a^3 - a|} \sum_{i=m}^{m+l-1} \left\| a^{3i+3} \left(f_o\left(\frac{a^2 x}{a^{i+2}}\right) - (a + a^3) f_o\left(\frac{ax}{a^{i+2}}\right) + a^4 f\left(\frac{x}{a^{i+2}}\right) \right) \right. \\ &\quad \left. - a^{i+1} \left(f_o\left(\frac{a^2 x}{a^{i+2}}\right) - (a + a^3) f\left(\frac{ax}{a^{i+2}}\right) + a^4 f_o\left(\frac{x}{a^{i+2}}\right) \right) \right\| \\ &\leqslant \sum_{i=m}^{m+l-1} \left(a^{2i} \mu\left(\frac{x}{a^{i+1}}\right) + \frac{|a^{3i+3} - a^{i+1}|}{|a^3 - a|} \nu\left(\frac{x}{a^{i+2}}\right) \right) \end{aligned} \tag{3.13}$$

for all $x \in V \setminus \{0\}$.

On account of (3.10) and (3.13), the sequence $\{J_m f(x)\}$ is a Cauchy sequence for all $x \in V \setminus \{0\}$. Since Y is complete and $f(0) = 0$, the sequence $\{J_m f(x)\}$ converges for all $x \in V$. Hence, we can define a mapping $F : V \to Y$ by

$$F(x) := \lim_{m\to\infty} \left(a^{3m+3} f_o^{(2)}\left(\frac{x}{a^{m+1}}\right) + a^{2m} f_e\left(\frac{x}{a^m}\right) + a^{m+1} f_o^{(1)}\left(\frac{x}{a^{m+1}}\right)\right)$$

for all $x \in V$. Moreover, if we put $m = 0$ and let $l \to \infty$ in (3.13), we obtain the inequality (3.12).

In view of the definition of F, (3.4), (3.11), and

$$\begin{aligned}
&DF(x_1, x_2, \dots, x_n) \\
&= \lim_{m\to\infty} \left(a^{2m} Df_e\left(\frac{x_1}{a^m}, \dots, \frac{x_n}{a^m}\right) + \frac{a^{3m+3} - a^{m+1}}{a^3 - a} Df_o\left(\frac{x_1}{a^m}, \dots, \frac{x_n}{a^m}\right)\right. \\
&\qquad\qquad \left. + \frac{a^{m+4} - a^{3m+4}}{a^3 - a} Df_o\left(\frac{x_1}{a^{m+1}}, \dots, \frac{x_n}{a^{m+1}}\right)\right)
\end{aligned}$$

for all $x_1, x_2, \dots, x_n \in V \setminus \{0\}$, we get the equalities in (3.6) for all $x \in V$ and we further obtain $DF(x_1, x_2, \dots, x_n) = 0$ for all $x_1, x_2, \dots, x_n \in V \setminus \{0\}$. We notice that the equalities in (3.9) hold in view of (3.6). Therefore, the equalities in (2.4) hold, for all $x \in V$, with $k = |a|$ if $|a| > 1$ or $k = \frac{1}{|a|}$ if $|a| < 1$.

When $|a| > 1$, according to Lemma 2.1, there exists a unique mapping $F : V \to Y$ satisfying the equalities in (3.6) and the inequality (3.12), since the inequality

$$\begin{aligned}
&\|f(x) - F(x)\| \\
&\leqslant \sum_{i=0}^{\infty} \left(a^{2i} \mu\left(\frac{x}{a^{i+1}}\right) + \frac{|a^{3i+3} - a^{i+1}|}{|a^3 - a|} \nu\left(\frac{x}{a^{i+2}}\right)\right) \\
&\leqslant \sum_{i=0}^{\infty} a^{2i} \left(\mu\left(\frac{x}{a^{i+1}}\right) + \mu\left(\frac{-x}{a^{i+1}}\right)\right) + \sum_{i=0}^{\infty} \frac{|a^{3i+3}|}{|a^3 - a|} \left(\nu\left(\frac{x}{a^{i+2}}\right) + \nu\left(\frac{-x}{a^{i+2}}\right)\right) \\
&= \sum_{i=0}^{\infty} k^{3i} \phi\left(\frac{x}{k^i}\right)
\end{aligned}$$

holds for all $x \in V \setminus \{0\}$, where $k := |a|$ and $\phi(x) := \mu\left(\frac{x}{a}\right) + \mu\left(\frac{-x}{a}\right) + \frac{|a^3|}{|a^3 - a|} \nu\left(\frac{x}{a^2}\right) + \frac{|a^3|}{|a^3 - a|} \nu\left(\frac{-x}{a^2}\right)$.

When $|a| < 1$, according to Lemma 2.1, there exists a unique mapping $F : V \to Y$ satisfying the equalities in (3.6) and the inequality (3.12), since the inequality

$$\begin{aligned}
&\|f(x) - F(x)\| \\
&\leqslant \sum_{i=0}^{\infty} \left(a^{2i} \mu\left(\frac{x}{a^{i+1}}\right) + \frac{|a^{3i+3} - a^{i+1}|}{|a^3 - a|} \nu\left(\frac{x}{a^{i+2}}\right)\right)
\end{aligned}$$

$$\leqslant a^2 \sum_{i=0}^{\infty} \left(\mu\left(\frac{x}{a^{i+1}}\right) + \mu\left(\frac{-x}{a^{i+1}}\right) \right) + \frac{|a^i|}{|a^3-a|} \sum_{i=0}^{\infty} \left(\nu\left(\frac{x}{a^{i+2}}\right) + \nu\left(\frac{-x}{a^{i+2}}\right) \right)$$
$$\leqslant \sum_{i=0}^{\infty} \frac{\phi(k^i x)}{k^i}$$

holds for all $x \in V \setminus \{0\}$, where $k := \frac{1}{|a|}$ and $\phi(x) := \mu\left(\frac{x}{a}\right) + \mu\left(\frac{-x}{a}\right) + \frac{1}{|a^3-a|}\nu\left(\frac{x}{a^2}\right) + \frac{1}{|a^3-a|}\nu\left(\frac{-x}{a^2}\right)$. □

We assume that μ, ν and φ satisfy different conditions from those of Theorems 3.1 and 3.2 and prove that there exists a unique exact solution near every approximate solution to $Df(x_1,x_2,\ldots,x_n) = 0$.

THEOREM 3.3. *Let a be a real constant with $a \notin \{-1,0,1\}$, let n be a fixed integer greater than 1, let $\mu : V \setminus \{0\} \to [0,\infty)$ be a function satisfying the conditions*

$$\begin{cases} \sum_{i=0}^{\infty} \frac{\mu(a^i x)}{|a|^{2i}} < \infty & \text{when } |a| > 1, \\ \sum_{i=0}^{\infty} |a|^{2i} \mu\left(\frac{x}{a^i}\right) < \infty & \text{when } |a| < 1 \end{cases} \tag{3.14}$$

for all $x \in V \setminus \{0\}$, let $\nu : V \setminus \{0\} \to [0,\infty)$ be a function satisfying the conditions

$$\begin{cases} \sum_{i=0}^{\infty} \frac{\nu(a^i x)}{|a|^{2i}} < \infty \text{ and } \sum_{i=0}^{\infty} |a|^i \nu\left(\frac{x}{a^i}\right) < \infty & \text{when } |a| > 1, \\ \sum_{i=0}^{\infty} \frac{\nu(a^i x)}{|a|^i} < \infty \text{ and } \sum_{i=0}^{\infty} |a|^{2i} \nu\left(\frac{x}{a^i}\right) < \infty & \text{when } |a| < 1 \end{cases} \tag{3.15}$$

for all $x \in V \setminus \{0\}$, and let $\varphi : (V \setminus \{0\})^n \to [0,\infty)$ be a function satisfying the conditions

$$\begin{cases} \sum_{i=0}^{\infty} \frac{\varphi(a^i x_1, a^i x_2, \ldots, a^i x_n)}{|a|^{2i}} < \infty \text{ and } \sum_{i=0}^{\infty} |a|^i \varphi\left(\frac{x_1}{a^i}, \frac{x_2}{a^i}, \ldots, \frac{x_n}{a^i}\right) < \infty \\ \qquad \text{when } |a| > 1, \\ \sum_{i=0}^{\infty} \frac{\varphi(a^i x_1, a^i x_2, \ldots, a^i x_n)}{|a|^i} < \infty \text{ and } \sum_{i=0}^{\infty} |a|^{2i} \varphi\left(\frac{x_1}{a^i}, \frac{x_2}{a^i}, \ldots, \frac{x_n}{a^i}\right) < \infty \\ \qquad \text{when } |a| < 1 \end{cases} \tag{3.16}$$

for all $x_1,x_2,\ldots,x_n \in V \setminus \{0\}$. If a mapping $f : V \to Y$ satisfies $f(0) = 0$ and the inequalities in (3.3) for all $x \in V \setminus \{0\}$ and (3.4) for all $x_1,x_2,\ldots,x_n \in V \setminus \{0\}$, then there exists a unique mapping $F : V \to Y$ satisfying the equality (3.5) for all

$x_1, x_2, \ldots, x_n \in V \setminus \{0\}$, *the equalities in* (3.6) *for all* $x \in V$, *and*

$$\|f(x) - F(x)\| \leqslant \begin{cases} \sum_{i=0}^{\infty} \frac{\mu(a^i x)}{a^{2i+2}} + \frac{1}{|a^3 - a|} \sum_{i=0}^{\infty} \left(\frac{\nu(a^i x)}{|a|^{3i+3}} + |a|^i \nu\left(\frac{x}{a^{i+1}}\right) \right) \\ \qquad \text{when } |a| > 1, \\ \sum_{i=0}^{\infty} |a|^{2i} \mu\left(\frac{x}{a^{i+1}}\right) + \frac{1}{|a^3 - a|} \sum_{i=0}^{\infty} \left(\frac{\nu(a^i x)}{|a|^{i+1}} + |a|^{3i} \nu\left(\frac{x}{a^{i+1}}\right) \right) \\ \qquad \text{when } |a| < 1 \end{cases} \tag{3.17}$$

for all $x \in V \setminus \{0\}$.

Proof. We will divide the proof of this theorem into two cases, the case for $|a| > 1$ and the other case for $|a| < 1$.

Case 1. Assume that $|a| > 1$. We define a set $A := \{f : V \to Y \mid f(0) = 0\}$ and a mapping $J_m : A \to A$ by

$$J_m f(x) := \frac{f_o^{(2)}(a^m x)}{a^{3m}} + \frac{f_e(a^m x)}{a^{2m}} + a^m f_o^{(1)}\left(\frac{x}{a^m}\right)$$

for all $x \in V$ and $m \in \mathbb{N}_0$. It follows from (3.3) that

$$\begin{aligned} &\|J_m f(x) - J_{m+l} f(x)\| \\ &\leqslant \sum_{i=m}^{m+l-1} \left\| \frac{f_e(a^{i+1}x) - a^2 f_e(a^i x)}{a^{2i+2}} \right\| \\ &\quad + \frac{1}{|a^3 - a|} \sum_{i=m}^{m+l-1} \left\| - \frac{f_o(a^2 \cdot a^i x) - (a + a^3) f_o(a^{i+1}x) + a^4 f(a^i x)}{a^{3i+3}} \right. \\ &\qquad \left. - a^i \left(f_o\left(\frac{a^2 x}{a^{i+1}}\right) - (a + a^3) f_o\left(\frac{ax}{a^{i+1}}\right) + a^4 f_o\left(\frac{x}{a^{i+1}}\right) \right) \right\| \\ &\leqslant \sum_{i=m}^{m+l-1} \frac{\mu(a^i x)}{a^{2i+2}} + \frac{1}{|a^3 - a|} \sum_{i=m}^{m+l-1} \left(\frac{\nu(a^i x)}{|a|^{3i+3}} + |a|^i \nu\left(\frac{x}{a^{i+1}}\right) \right) \end{aligned} \tag{3.18}$$

for all $x \in V \setminus \{0\}$.

In view of (3.14), (3.15), and (3.18), the sequence $\{J_m f(x)\}$ is a Cauchy sequence for all $x \in V \setminus \{0\}$. Since Y is complete and $f(0) = 0$, the sequence $\{J_m f(x)\}$ converges for all $x \in V$. Hence, we can define a mapping $F : V \to Y$ by

$$F(x) := \lim_{m \to \infty} \frac{f_o^{(2)}(a^m x)}{a^{3m}} + \frac{f_e(a^m x)}{a^{2m}} + a^m f_o^{(1)}\left(\frac{x}{a^m}\right)$$

for all $x \in V$. Moreover, if we put $m = 0$ and let $l \to \infty$ in (3.18), we obtain the first inequality of (3.17).

Using the definition of F, (3.4), (3.16), and

$$
\begin{aligned}
&DF(x_1,x_2,\ldots,x_n)\\
&= \lim_{m\to\infty}\Bigg(\frac{Df_o\big(a^{m+1}x_1,\ldots,a^{m+1}x_n\big)-aDf_o\big(a^m x_1,\ldots,a^m x_n\big)}{a^{3m}(a^3-a)}\\
&\qquad\qquad +\frac{Df_e\big(a^m x_1,\ldots,a^m x_n\big)}{a^{2m}}\\
&\qquad\qquad -\frac{a^m}{(a^3-a)}\left(Df_o\left(\frac{ax_1}{a^m},\ldots,\frac{ax_n}{a^m}\right)-a^3Df_o\left(\frac{x_1}{a^m},\ldots,\frac{x_n}{a^m}\right)\right)\Bigg)
\end{aligned}
$$

for all $x_1,x_2,\ldots,x_n\in V\setminus\{0\}$, we get the equalities in (3.6) for all $x\in V$ and we further get $DF(x_1,x_2,\ldots,x_n)=0$ for all $x_1,x_2,\ldots,x_n\in V\setminus\{0\}$. We notice that the equalities in (2.4) are true in view of (3.6), where $k=|a|$.

Using Lemma 2.2, we conclude that there exists a unique mapping $F:V\to Y$ satisfying the equalities in (3.6) and the first inequality in (3.17), since the inequality

$$
\begin{aligned}
\|f(x)-F(x)\| &\leqslant \sum_{i=0}^{\infty}\frac{\mu(a^i x)}{a^{2i+2}}+\frac{1}{|a^3-a|}\sum_{i=0}^{\infty}\left(\frac{\nu(a^i x)}{|a|^{3i+3}}+|a|^i\nu\left(\frac{x}{a^{i+1}}\right)\right)\\
&\leqslant \sum_{i=0}^{\infty}\frac{\mu(a^i x)}{a^{2i}}+\frac{1}{|a^3-a|}\sum_{i=0}^{\infty}\left(\frac{\nu(a^i x)}{|a|^{2i}}+|a|^i\nu\left(\frac{x}{a^{i+1}}\right)\right)\\
&\leqslant \sum_{i=0}^{\infty}\left(\frac{\psi(k^i x)}{k^{2i}}+k^i\phi\left(\frac{x}{k^i}\right)\right)
\end{aligned}
$$

holds for all $x\in V\setminus\{0\}$, where $k:=|a|$, $\phi(x):=\frac{\nu(\frac{x}{a})+\nu(\frac{-x}{a})}{|a^3-a|}$, and $\psi(x):=\mu(x)+\mu(-x)+\frac{\nu(x)+\nu(-x)}{|a^3-a||a|^3}$.

Case 2. We now consider the case of $|a|<1$ and define a mapping $J_m:A\to A$ by

$$
J_m f(x):=a^{3m}f_o^{(2)}\left(\frac{x}{a^m}\right)+a^{2m}f_e\left(\frac{x}{a^m}\right)+\frac{f_o^{(1)}(a^m x)}{a^m}
$$

for all $x\in V$ and $n\in\mathbb{N}_0$. It follows from (3.3) that

$$
\begin{aligned}
&\|J_m f(x)-J_{m+l}f(x)\|\\
&\leqslant \sum_{i=m}^{m+l-1}a^{2i}\left\|f_e\left(a\frac{x}{a^{i+1}}\right)-a^2f_e\left(\frac{x}{a^{i+1}}\right)\right\|\\
&\quad+\frac{1}{|a^3-a|}\sum_{i=m}^{m+l-1}\Bigg\|a^{3i}\left(f_o\left(\frac{a^2x}{a^{i+1}}\right)-(a+a^3)f_o\left(\frac{ax}{a^{i+1}}\right)+a^4f_o\left(\frac{x}{a^{i+1}}\right)\right)\\
&\qquad\qquad+\frac{f_o(a^2a^ix)-(a+a^3)f_o(a^{i+1}x)+a^4f_o(a^ix)}{a^{i+1}}\Bigg\| \qquad (3.19)\\
&\leqslant \sum_{i=m}^{m+l-1}|a|^{2i}\mu\left(\frac{x}{a^{i+1}}\right)+\frac{1}{|a^3-a|}\sum_{i=m}^{m+l-1}\left(\frac{\nu(a^ix)}{|a|^{i+1}}+|a|^{3i}\nu\left(\frac{x}{a^{i+1}}\right)\right)
\end{aligned}
$$

for all $x \in V \setminus \{0\}$.

On account of (3.14), (3.15), and (3.19), the sequence $\{J_m f(x)\}$ is a Cauchy sequence for all $x \in V \setminus \{0\}$. Since Y is complete and $f(0) = 0$, the sequence $\{J_m f(x)\}$ converges for all $x \in V$. Hence, we can define a mapping $F : V \to Y$ by

$$F(x) := \lim_{m\to\infty} \left(a^{3m} f_o^{(2)}\left(\frac{x}{a^m}\right) + a^{2m} f_e\left(\frac{x}{a^m}\right) + \frac{f_o^{(1)}(a^m x)}{a^m} \right)$$

for all $x \in V$. Moreover, if we put $m = 0$ and let $l \to \infty$ in (3.19), we obtain the second inequality in (3.17).

By the definition of F, (3.4), (3.16), and

$$\begin{aligned} &DF(x_1, x_2, \dots, x_n) \\ &= \lim_{m\to\infty} \Bigg(a^{2m} Df_e\left(\frac{x_1}{a^m}, \dots, \frac{x_n}{a^m}\right) \\ &\qquad + \frac{a^{3m}}{a^3 - a} Df_o\left(\frac{x_1}{a^{m-1}}, \dots, \frac{x_n}{a^{m-1}}\right) - \frac{a^{3m+1}}{a^3 - a} Df_o\left(\frac{x_1}{a^m}, \dots, \frac{x_n}{a^m}\right) \\ &\qquad - \frac{Df_o(a^{m+1}x_1, \dots, a^{m+1}x_n) - a^3 Df_o(a^m x_1, \dots, a^m x_n)}{a^m(a^3 - a)} \Bigg) \end{aligned}$$

for all $x_1, x_2, \dots, x_n \in V \setminus \{0\}$, we get the equalities in (3.6) for all $x \in V$ and we moreover obtain $DF(x_1, x_2, \dots, x_n) = 0$ for all $x_1, x_2, \dots, x_n \in V \setminus \{0\}$. We remark that the equalities in (2.4) hold by considering (3.6) with $k = \frac{1}{|a|}$.

Using Lemma 2.2, we conclude that there exists a unique mapping $F : V \to Y$ satisfying the equalities in (3.6) and the second inequality in (3.17), since the inequality

$$\begin{aligned} \|f(x) - F(x)\| &\leqslant \sum_{i=0}^{\infty} |a|^{2i} \mu\left(\frac{x}{a^{i+1}}\right) + \frac{1}{|a^3 - a|} \sum_{i=0}^{\infty} \left(\frac{\nu(a^i x)}{|a|^{i+1}} + |a|^{3i} \nu\left(\frac{x}{a^{i+1}}\right) \right) \\ &= \sum_{i=0}^{\infty} \Bigg(\frac{\nu(a^i x) + \nu(-a^i x)}{|a^3 - a||a|^{i+1}} + |a|^{2i} \left(\mu\left(\frac{x}{a^{i+1}}\right) + \mu\left(\frac{-x}{a^{i+1}}\right) \right) \\ &\qquad + \frac{|a|^{2i}}{|a^3 - a|} \left(\nu\left(\frac{x}{a^{i+1}}\right) + \nu\left(\frac{-x}{a^{i+1}}\right) \right) \Bigg) \\ &\leqslant \sum_{i=0}^{\infty} \left(\frac{\psi(k^i x)}{k^{2i}} + k^i \phi\left(\frac{x}{k^i}\right) \right) \end{aligned}$$

holds for all $x \in V \setminus \{0\}$, where $k := \frac{1}{|a|}$, $\phi(x) := \frac{1}{|a^3 - a||a|}\big(\nu(x) + \nu(-x)\big)$, and $\psi(x) := \frac{1}{|a^3 - a|}\big(\nu\big(\frac{x}{a}\big) + \nu\big(\frac{-x}{a}\big)\big) + \mu\big(\frac{x}{a}\big) + \mu\big(\frac{-x}{a}\big)$. $\square$

Suppose μ, ν and φ satisfy other conditions from those of preceding three theorems. In the following theorem, we prove that there exists a unique exact solution near every approximate solution to $Df(x_1, x_2, \dots, x_n) = 0$.

THEOREM 3.4. *Suppose a is a real constant with $a \notin \{-1,0,1\}$. Let n be a fixed integer greater than 1, let $\mu : V \setminus \{0\} \to [0,\infty)$ be a function satisfying the conditions*

$$\begin{cases} \sum_{i=0}^{\infty} |a|^{2i} \mu\left(\frac{x}{a^i}\right) < \infty \text{ when } |a| > 1, \\ \sum_{i=0}^{\infty} \frac{\mu(a^i x)}{|a|^{2i}} < \infty \qquad \text{when } |a| < 1 \end{cases} \tag{3.20}$$

for all $x \in V \setminus \{0\}$, let $\nu : V \setminus \{0\} \to [0,\infty)$ be a function satisfying the conditions

$$\begin{cases} \sum_{i=0}^{\infty} \frac{\nu(a^i x)}{|a|^{3i}} < \infty \text{ and } \sum_{i=0}^{\infty} |a|^{2i} \nu\left(\frac{x}{a^i}\right) < \infty \text{ when } |a| > 1, \\ \sum_{i=0}^{\infty} \frac{\nu(a^i x)}{|a|^{2i}} < \infty \text{ and } \sum_{i=0}^{\infty} |a|^{3i} \nu\left(\frac{x}{a^i}\right) < \infty \text{ when } |a| < 1 \end{cases} \tag{3.21}$$

for all $x \in V \setminus \{0\}$, and let $\varphi : (V \setminus \{0\})^n \to [0,\infty)$ be a function satisfying the conditions

$$\begin{cases} \sum_{i=0}^{\infty} \frac{\varphi(a^i x_1, a^i x_2, \ldots, a^i x_n)}{|a|^{3i}} < \infty \text{ and } \sum_{i=0}^{\infty} |a|^{2i} \varphi\left(\frac{x_1}{a^i}, \frac{x_2}{a^i}, \ldots, \frac{x_n}{a^i}\right) < \infty \\ \qquad \text{when } |a| > 1, \\ \sum_{i=0}^{\infty} \frac{\varphi(a^i x_1, a^i x_2, \ldots, a^i x_n)}{|a|^{2i}} < \infty \text{ and } \sum_{i=0}^{\infty} |a|^{3i} \varphi\left(\frac{x_1}{a^i}, \frac{x_2}{a^i}, \ldots, \frac{x_n}{a^i}\right) < \infty \\ \qquad \text{when } |a| < 1 \end{cases} \tag{3.22}$$

for all $x_1, x_2, \ldots, x_n \in V \setminus \{0\}$. If a mapping $f : V \to Y$ satisfies $f(0) = 0$ and the inequalities in (3.3) for all $x \in V \setminus \{0\}$ and (3.4) for all $x_1, x_2, \ldots, x_n \in V \setminus \{0\}$, then there exists a unique mapping $F : V \to Y$ satisfying the equality (3.5) for all $x_1, x_2, \ldots, x_n \in V \setminus \{0\}$, the equalities in (3.6) for all $x \in V$, and

$$\|f(x) - F(x)\| \leqslant \begin{cases} |a|^{2i} \mu\left(\frac{x}{a^{i+1}}\right) + \frac{1}{|a^3 - a|}\left(\frac{\nu(a^i x)}{|a|^{3i+3}} + |a|^i \nu\left(\frac{x}{a^{i+1}}\right)\right) \text{ when } |a| > 1, \\ \frac{\mu(a^i x)}{|a|^{2i+2}} + \frac{1}{|a^3 - a|}\left(\frac{\nu(a^i x)}{|a|^{i+1}} + |a|^{3i} \nu\left(\frac{x}{a^{i+1}}\right)\right) \qquad \text{when } |a| < 1 \end{cases} \tag{3.23}$$

for all $x \in V \setminus \{0\}$.

Proof. We will divide the proof of this theorem into two cases; namely, the case for $|a| > 1$ and the other case for $|a| < 1$.

Case 1. Assume that $|a| > 1$. We define a set $A := \{f : V \to Y \,|\, f(0) = 0\}$ and a mapping $J_m : A \to A$ by

$$J_m f(x) := \frac{f_o^{(2)}(a^m x)}{a^{3m}} + a^{2m} f_e\left(\frac{x}{a^m}\right) + a^m f_o^{(1)}\left(\frac{x}{a^m}\right)$$

for all $x \in V$ and $m \in \mathbb{N}_0$. It follows from (3.3) that

$$\begin{aligned} &\|J_m f(x) - J_{m+l} f(x)\| \\ &\leqslant \sum_{i=m}^{m+l-1} \left\| a^{2i}\left(f_e\left(\frac{ax}{a^{i+1}}\right) - a^2 f_e\left(\frac{x}{a^{i+1}}\right)\right)\right\| \\ &\quad + \frac{1}{|a^3-a|} \sum_{i=m}^{m+l-1} \left\| -\frac{f_o(a^2a^ix) - (a+a^3)f_o(a^{i+1}x) + a^4 f(a^i x)}{a^{3i+3}} \right. \\ &\qquad\qquad \left. - a^i\left(f_o\left(\frac{a^2x}{a^{i+1}}\right) - (a+a^3) f_o\left(\frac{ax}{a^{i+1}}\right) + a^4 f\left(\frac{x}{a^{i+1}}\right)\right)\right\| \\ &\leqslant \sum_{i=m}^{m+l-1} \left(|a|^{2i}\mu\left(\frac{x}{a^{i+1}}\right) + \frac{1}{|a^3-a|}\left(\frac{\nu(a^i x)}{|a|^{3i+3}} + |a|^i \nu\left(\frac{x}{a^{i+1}}\right)\right)\right) \end{aligned} \tag{3.24}$$

for all $x \in V \setminus \{0\}$.

In view of (3.20), (3.21), and (3.24), the sequence $\{J_m f(x)\}$ is a Cauchy sequence for all $x \in V \setminus \{0\}$. Since Y is complete and $f(0) = 0$, the sequence $\{J_m f(x)\}$ converges for all $x \in V$. Hence, we can define a mapping $F : V \to Y$ by

$$F(x) := \lim_{m\to\infty} \left(\frac{f_o^{(2)}(a^m x)}{a^{3m}} + a^{2m} f_e\left(\frac{x}{a^m}\right) + a^m f_o^{(1)}\left(\frac{x}{a^m}\right)\right)$$

for all $x \in V$. Moreover, if we put $m = 0$ and let $l \to \infty$ in (3.24), we obtain the first inequality of (3.23).

Using the definition of F, (3.4), (3.22), and

$$\begin{aligned} &DF(x_1, x_2, \ldots, x_n) \\ &= \lim_{m\to\infty} \left(\frac{Df_o(a^{m+1}x_1, \ldots, a^{m+1}x_n) - aDf_o(a^m x_1, \ldots, a^m x_n)}{a^{3m}(a^3-a)} + a^{2m} Df_e\left(\frac{x_1}{a^m}, \ldots, \frac{x_n}{a^m}\right)\right. \\ &\qquad \left. + \frac{a^m}{(a^3-a)}\left(Df_o\left(\frac{x_1}{a^{m-1}}, \ldots, \frac{x_n}{a^{m-1}}\right) - a^3 Df_o\left(\frac{x_1}{a^m}, \ldots, \frac{x_n}{a^m}\right)\right)\right) \end{aligned}$$

for all $x_1, x_2, \ldots, x_n \in V \setminus \{0\}$, we obtain the equalities in (3.6) for all $x \in V$ and we further get $DF(x_1, x_2, \ldots, x_n) = 0$ for all $x_1, x_2, \ldots, x_n \in V \setminus \{0\}$. We notice that the equalities in (2.4) are true in view of (3.6), where $k = |a|$.

Using Lemma 2.3, we conclude that there exists a unique mapping $F : V \to Y$ satisfying the equalities in (3.6) and the first inequality in (3.23), since the inequality

$$\begin{aligned} &\|f(x) - F(x)\| \\ &\leqslant \sum_{i=0}^{\infty} \left(|a|^{2i}\mu\left(\frac{x}{a^{i+1}}\right) + \frac{1}{|a^3-a|}\left(\frac{\nu(a^i x)}{|a|^{3i+3}} + |a|^i \nu\left(\frac{x}{a^{i+1}}\right)\right)\right) \end{aligned}$$

$$\leqslant \sum_{i=0}^{\infty}\Bigg(|a|^{2i}\mu\left(\frac{x}{a^{i+1}}\right)+|a|^{2i}\mu\left(\frac{-x}{a^{i+1}}\right)+\frac{|a|^{2i}}{|a^3-a|}\left(\nu\left(\frac{x}{a^{i+1}}\right)+\nu\left(\frac{-x}{a^{i+1}}\right)\right)$$
$$+\frac{1}{|a^3-a|}\left(\frac{\nu(a^ix)}{|a|^{3i+3}}+\frac{\nu(-a^ix)}{|a|^{3i+3}}\right)\Bigg)$$
$$\leqslant \sum_{i=0}^{\infty}\left(\frac{\psi(k^ix)}{k^{3i}}+k^{2i}\phi\left(\frac{x}{k^i}\right)\right)$$

holds for all $x\in V\setminus\{0\}$, where $k:=|a|$, $\phi(x):=\mu\left(\frac{x}{a}\right)+\mu\left(\frac{-x}{a}\right)+\frac{\nu(\frac{x}{a})+\nu(\frac{-x}{a})}{|a^3-a|}$, and $\psi(x):=\frac{\nu(x)+\nu(-x)}{|a^3-a||a|^3}$.

Case 2. We now consider the case of $|a|<1$ and define a mapping $J_m : A\to A$ by

$$J_mf(x):=a^{3m}f_o^{(2)}\left(\frac{x}{a^m}\right)+\frac{f_e(a^mx)}{a^{2m}}+\frac{f_o^{(1)}(a^mx)}{a^m}$$

for all $x\in V$ and $n\in\mathbb{N}_0$. It follows from (3.3) that

$$\begin{aligned}
&\|J_mf(x)-J_{m+l}f(x)\|\\
&\leqslant \sum_{i=m}^{m+l-1}\left\|\frac{a^2f_e(a^ix)-f_e(a^iax)}{a^{2i+2}}\right\|\\
&\quad+\frac{1}{|a^3-a|}\sum_{i=m}^{m+l-1}\Bigg\|a^{3i}\left(f_o\left(\frac{a^2x}{a^{i+1}}\right)-(a+a^3)f_o\left(\frac{ax}{a^{i+1}}\right)+a^4f_o\left(\frac{x}{a^{i+1}}\right)\right)\\
&\qquad+\frac{f_o(a^2a^ix)-(a+a^3)f_o(a^{i+1}x)+a^4f_o(a^ix)}{a^{i+1}}\Bigg\|\\
&\leqslant \sum_{i=m}^{m+l-1}\left(\frac{\mu(a^ix)}{|a|^{2i+2}}+\frac{1}{|a^3-a|}\left(\frac{\nu(a^ix)}{|a|^{i+1}}+|a|^{3i}\nu\left(\frac{x}{a^{i+1}}\right)\right)\right)
\end{aligned}\tag{3.25}$$

for all $x\in V\setminus\{0\}$.

On account of (3.20), (3.21) and (3.25), the sequence $\{J_mf(x)\}$ is a Cauchy sequence for all $x\in V\setminus\{0\}$. Since Y is complete and $f(0)=0$, the sequence $\{J_mf(x)\}$ converges for all $x\in V$. Hence, we can define a mapping $F:V\to Y$ by

$$F(x):=\lim_{m\to\infty}a^{3m}f_o^{(2)}\left(\frac{x}{a^m}\right)+\frac{f_e(a^mx)}{a^{2m}}+\frac{f_o^{(1)}(a^mx)}{a^m}$$

for all $x\in V$. Moreover, if we put $m=0$ and let $l\to\infty$ in (3.25), we obtain the second inequality in (3.23).

By the definition of F, (3.4), (3.22), and

$$\begin{aligned}
&DF(x_1,x_2,\dots,x_n)\\
&= \lim_{m\to\infty}\left(\frac{a^{3m}}{a^3-a}Df_o\left(\frac{x_1}{a^{m-1}},\dots,\frac{x_n}{a^{m-1}}\right)-\frac{a^{3m+1}}{a^3-a}Df_o\left(\frac{x_1}{a^m},\dots,\frac{x_n}{a^m}\right)\right.\\
&\qquad\qquad+\frac{Df_e\big(a^m x_1,\dots,a^m x_n\big)}{a^{2m}}\\
&\qquad\qquad\left.-\frac{Df_o\big(a^{m+1}x_1,\dots,a^{m+1}x_n\big)-a^3 Df_o\big(a^m x_1,\dots,a^m x_n\big)}{a^m(a^3-a)}\right)
\end{aligned}$$

for all $x_1,x_2,\dots,x_n\in V\setminus\{0\}$, we get the equalities in (3.6) for all $x\in V$ and we moreover have $DF(x_1,x_2,\dots,x_n)=0$ for all $x_1,x_2,\dots,x_n\in V\setminus\{0\}$. We remark that the equalities in (2.4) hold by considering (3.6) with $k=\frac{1}{|a|}$.

Using Lemma 2.3, we conclude that there exists a unique mapping $F:V\to Y$ satisfying the equalities in (3.6) and the second inequality in (3.23), since the inequality

$$\begin{aligned}
&\|f(x)-F(x)\|\\
&\leqslant \sum_{i=0}^{\infty}\left(\frac{\mu(a^i x)}{|a|^{2i+2}}+\frac{1}{|a^3-a|}\left(\frac{\nu(a^i x)}{|a|^{i+1}}+|a|^{3i}\nu\left(\frac{x}{a^{i+1}}\right)\right)\right)\\
&=\sum_{i=0}^{\infty}\left(\frac{\mu(a^i x)+\mu(-a^i x)}{|a|^{2i+2}}+\frac{\nu(a^i x)+\nu(-a^i x)}{|a^3-a||a|^{2i+1}}+\frac{|a|^{3i}}{|a^3-a|}\left(\nu\left(\frac{x}{a^{i+1}}\right)+\nu\left(-\frac{x}{a^{i+1}}\right)\right)\right)\\
&\leqslant \sum_{i=0}^{\infty}\left(\frac{\psi(k^i x)}{k^{3i}}+k^{2i}\phi\left(\frac{x}{k^i}\right)\right)
\end{aligned}$$

holds for all $x\in V\setminus\{0\}$, where $k:=\frac{1}{|a|}$, $\phi(x):=\frac{1}{|a|^2}\big(\mu(x)+\mu(-x)\big)+\frac{1}{|a^3-a||a|}\big(\nu(x)+\nu(-x)\big)$, and $\psi(x):=\frac{1}{|a^3-a|}\big(\mu\big(\frac{x}{a}\big)+\mu\big(-\frac{x}{a}\big)\big)$. □

By using Theorems 3.1, 3.2, 3.3, and 3.4, we can prove the following corollary.

COROLLARY 3.5. *Let X be a normed space and let p, ε, θ, ξ be real constants such that $p\notin\{1,2,3\}$, $a\notin\{-1,0,1\}$, $\xi>0$, and $\theta>0$. If a mapping $f:X\to Y$ satisfies $f(0)=0$,*

$$\big\|f_e(ax)-a^2 f_e(x)\big\|\leqslant\varepsilon\|x\|^p, \tag{3.26}$$

and

$$\big\|f_o(a^2x)-(a+a^3)f_o(ax)+a^4 f_o(x)\big\|\leqslant\theta\|x\|^p \tag{3.27}$$

for all $x\in X\setminus\{0\}$, as well as if f satisfies the inequality

$$\|Df(x_1,x_2,\dots,x_n)\|\leqslant\xi\big(\|x_1\|^p+\dots+\|x_n\|^p\big) \tag{3.28}$$

for all $x_1, x_2, \ldots, x_n \in X \setminus \{0\}$, *then there exists a unique mapping* $F : X \to Y$ *satisfying* (3.5) *for all* $x_1, x_2, \ldots, x_n \in X \setminus \{0\}$, (3.6) *for all* $x \in X$, *as well as*

$$\|f(x) - F(x)\| \leqslant \begin{cases} \left(\dfrac{|a|}{||a| - |a|^p|} + \dfrac{|a|^3}{||a|^3 - |a|^p|}\right) \dfrac{\theta \|x\|^p}{|a|^p |a^3 - a|} + \dfrac{\varepsilon \|x\|^p}{|a^2 - |a|^p|} \\ \qquad \text{when either } |a| \text{ and } p < 1 \text{ or } |a| > 1 \text{ and } p > 3, \\ \left(\dfrac{1}{||a| - |a|^p|} + \dfrac{1}{||a|^3 - |a|^p|}\right) \dfrac{\theta \|x\|^p}{|a^3 - a|} + \dfrac{\varepsilon \|x\|^p}{|a^2 - |a|^p|} \\ \qquad \text{for the other cases} \end{cases} \tag{3.29}$$

for all $x \in X \setminus \{0\}$.

Proof. Let us put $\mu(x) := \varepsilon \|x\|^p$, $\nu(x) := \theta \|x\|^p$, and $\varphi(x_1, x_2, \ldots, x_n) := \theta\big(\|x_1\|^p + \cdots + \|x_n\|^p\big)$ for all $x_1, x_2, \ldots, x_n \in X \setminus \{0\}$. Then φ, μ, ν satisfy (3.1) and (3.2) when either $|a| > 1$ and $p < 1$ or when $|a| < 1$ and $p > 3$. If either $|a| > 1$ and $p > 3$ or if $|a| < 1$ and $p < 1$, then φ, μ, ν satisfy (3.10) and (3.11). Moreover, φ, μ, ν satisfy (3.14), (3.15), and (3.16) when $1 < p < 2$ and φ, μ, ν satisfy (3.20), (3.21), and (3.22) when $2 < p < 3$. Therefore, by Theorems 3.1, 3.2, 3.3, and 3.4, there exists a unique mapping $F : X \to Y$ such that (3.5) holds for all $x_1, x_2, \ldots, x_n \in X \setminus \{0\}$, and (3.6) holds for all $x \in X$, and such that (3.29) holds for all $x \in X \setminus \{0\}$. □

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