

EQUIVALENT PROPERTIES OF A HILBERT-TYPE INTEGRAL INEQUALITY WITH THE BEST CONSTANT FACTOR RELATED TO THE HURWITZ ZETA FUNCTION

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ABSTRACT. By the use of methods of real analysis and weight functions, we study the equivalent properties of a Hilbert-type integral inequality with the non-homogeneous kernel. The constant factor related to the Hurwitz zeta function is proved to be the best possible. As a corollary, a few equivalent conditions of a Hilbert-type integral inequality with the homogeneous kernel are deduced. We also consider their operator expressions.

1. INTRODUCTION

In 1925, by introducing one pair of conjugate exponents (p, q) , Hardy [1] proved the following integral inequality: For

$$p > 1, \frac{1}{p} + \frac{1}{q} = 1, f(x), g(y) \geq 0, 0 < \int_0^\infty f^p(x) dx < \infty \text{ and } 0 < \int_0^\infty g^q(y) dy < \infty,$$

we have the following Hardy-Hilbert's inequality

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} dx dy < \frac{\pi}{\sin(\pi/p)} \left(\int_0^\infty f^p(x) dx \right)^{\frac{1}{p}} \left(\int_0^\infty g^q(y) dy \right)^{\frac{1}{q}}, \quad (1.1)$$

where, the constant factor $\frac{\pi}{\sin(\pi/p)}$ is the best possible.

For $p = q = 2$, (1.1) reduces to the well known Hilbert integral inequality. Inequality (1.1) as well as Hilbert's integral inequality are important in mathematical analysis and its applications (cf. [2], [3]).

In 1934, Hardy et al. presented an extension of (1.1) as follows: If $k_1(x, y)$ is a non-negative homogeneous function of degree -1 ,

$$k_p = \int_0^\infty k_1(u, 1) u^{-\frac{1}{p}} du \in \mathbf{R}_+ = (0, \infty),$$

then we have the following Hardy-Hilbert-type integral inequality:

$$\int_0^\infty \int_0^\infty k_1(x, y) f(x) g(y) dx dy < k_p \left(\int_0^\infty f^p(x) dx \right)^{\frac{1}{p}} \left(\int_0^\infty g^q(y) dy \right)^{\frac{1}{q}}, \quad (1.2)$$

where the constant factor k_p is the best possible (cf. [2], Theorem 319). Additionally, the following Hilbert-type integral inequality with the non-homogeneous

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kernel was proved:

If $h(u) > 0$, $\phi(\sigma) = \int_0^\infty h(u)u^{\sigma-1}du \in \mathbf{R}_+$, then

$$\begin{aligned} & \int_0^\infty \int_0^\infty h(xy)f(x)g(y)dx dy \\ & < \phi\left(\frac{1}{p}\right) \left(\int_0^\infty x^{p-2}f^p(x)dx\right)^{\frac{1}{p}} \left(\int_0^\infty g^q(y)dy\right)^{\frac{1}{q}}, \end{aligned} \quad (1.3)$$

where the constant factor $\phi\left(\frac{1}{p}\right)$ is the best possible (cf. [2], Theorem 350).

In 1998, by introducing an independent parameter $\lambda > 0$, Yang extended Hilbert's integral inequality (cf. [4], [5]) proving that:

$$\begin{aligned} & \int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x+y)^\lambda} dx dy \\ & < B\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right) \left(\int_0^\infty x^{1-\lambda}f^2(x)dx\right)^{\frac{1}{2}} \left(\int_0^\infty y^{1-\lambda}g^2(y)dy\right)^{\frac{1}{2}}, \end{aligned} \quad (1.4)$$

where the constant factor $B\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right)$ is the best possible ($B(u, v)$ is the beta function).

In 2004, by introducing another pair of conjugate exponents (r, s) , Yang [6] presented the following extension of (1.1):

If $\lambda > 0$, $r > 1$, $f(x), g(y) \geq 0$,

$$\frac{1}{r} + \frac{1}{s} = 1, \quad 0 < \int_0^\infty x^{p(1-\frac{\lambda}{r})-1}f^p(x)dx < \infty \quad \text{and} \quad 0 < \int_0^\infty y^{q(1-\frac{\lambda}{s})-1}g^q(y)dy < \infty,$$

then

$$\begin{aligned} & \int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x^\lambda + y^\lambda} dx dy \\ & < \frac{\pi}{\lambda \sin(\pi/r)} \left[\int_0^\infty x^{p(1-\frac{\lambda}{r})-1}f^p(x)dx\right]^{\frac{1}{p}} \left[\int_0^\infty y^{q(1-\frac{\lambda}{s})-1}g^q(y)dy\right]^{\frac{1}{q}}, \end{aligned} \quad (1.5)$$

where the constant factor $\frac{\pi}{\lambda \sin(\pi/r)}$ is the best possible.

For $\lambda = 1, r = q, s = p$, (1.5) reduces to (1.1); For $\lambda = 1, r = p, s = q$, (1.5) reduces to the dual form of (1.1), namely:

$$\begin{aligned} & \int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} dx dy \\ & < \frac{\pi}{\sin(\pi/p)} \left(\int_0^\infty x^{p-2}f^p(x)dx\right)^{\frac{1}{p}} \left(\int_0^\infty y^{q-2}g^q(y)dy\right)^{\frac{1}{q}}, \end{aligned} \quad (1.6)$$

where the constant factor $\frac{\pi}{\sin(\pi/p)}$ is still the best possible.

In 2005, in the paper [7] an extension of (1.1) and (1.4) was also proved with the kernel $\frac{1}{(x+y)^\lambda}$ and two pairs of conjugate exponents. Krnić et al. [8]-[13] provided some extensions and particular cases of (1.1), (1.2) and (1.3) with parameters.

In 2009, Yang gave an extension of (1.2), (1.4) and (1.5) as follows (cf. [14], [15]): If $\lambda_1 + \lambda_2 = \lambda \in \mathbf{R} = (-\infty, \infty)$, $k_\lambda(x, y)$ is a non-negative homogeneous

function of degree $-\lambda$, satisfying

$$k_\lambda(ux, uy) = u^{-\lambda} k_\lambda(x, y) (u, x, y > 0),$$

and

$$k(\lambda_1) = \int_0^\infty k_\lambda(u, 1) u^{\lambda_1-1} du \in \mathbf{R}_+ = (0, \infty),$$

then we have

$$\begin{aligned} & \int_0^\infty \int_0^\infty k_\lambda(x, y) f(x) g(y) dx dy \\ & < k(\lambda_1) \left(\int_0^\infty x^{p(1-\lambda_1)-1} f^p(x) dx \right)^{\frac{1}{p}} \left(\int_0^\infty y^{q(1-\lambda_2)-1} g^q(y) dy \right)^{\frac{1}{q}}, \end{aligned} \quad (1.7)$$

where the constant factor $k(\lambda_1)$ is the best possible.

For $\lambda = 1$, $\lambda_1 = \frac{1}{q}$, $\lambda_2 = \frac{1}{p}$, (1.7) reduces to (1.2); for

$$p = q = 2, \lambda_1 = \lambda_2 = \frac{\lambda}{2} > 0, k_\lambda(x, y) = \frac{1}{(x+y)^\lambda},$$

(1.7) reduces to (1.4); for

$$\lambda > 0, \lambda_1 = \frac{\lambda}{r}, \lambda_2 = \frac{\lambda}{s}, k_\lambda(x, y) = \frac{1}{x^\lambda + y^\lambda},$$

(1.7) reduces to (1.5).

Moreover, the following extension of (1.3) was proved (cf. [15], [16]):

$$\begin{aligned} & \int_0^\infty \int_0^\infty h(xy) f(x) g(y) dx dy \\ & < \phi(\sigma) \left(\int_0^\infty x^{p(1-\sigma)-1} f^p(x) dx \right)^{\frac{1}{p}} \left(\int_0^\infty y^{q(1-\sigma)-1} g^q(y) dy \right)^{\frac{1}{q}}, \end{aligned} \quad (1.8)$$

where the constant factor $\phi(\sigma)$ is the best possible (cf. [16]).

For $\sigma = \frac{1}{p}$, (1.8) reduces to (1.3).

Some equivalent inequalities of (1.7) and (1.8) were established in [15]. In 2013, Yang [16] also studied the equivalency between (1.7) and (1.8) under the additional condition $h(u) = k_\lambda(u, 1)$. In 2017, Hong [17] considered an equivalent condition between (1.7) and a few parameters.

In the present paper, by the use of methods of real analysis and weight functions, we study the equivalent properties of a Hilbert-type integral inequality with the non-homogeneous kernel

$$e^{\alpha xy} \operatorname{csc} h(xy) \quad (\alpha < 1).$$

The constant factor related to the Hurwitz zeta function is proved to be the best possible. As a corollary, a few equivalent conditions of a Hilbert-type integral inequality with the homogeneous kernel are deduced. We also consider their operator expressions.

2. AN EXAMPLE AND A LEMMA

Example 2.1. For $\alpha < 1$, we set

$$h(u) := e^{\alpha u} \operatorname{csc} h(u) = \frac{2e^{\alpha u}}{e^u - e^{-u}} \quad (u > 0),$$

where

$$\operatorname{csc} h(u) = \frac{2}{e^u - e^{-u}}$$

is the hyperbolic cosecant function (cf. [18]).

For $\sigma > 1$, by the Lebesgue term by term integration theorem (cf. [19]), we derive that

$$\begin{aligned} k(\sigma, \alpha) &: = \int_0^\infty e^{\alpha u} \operatorname{csc} h(u) u^{\sigma-1} du \\ &= \int_0^\infty \frac{2e^{\alpha u} u^{\sigma-1}}{e^u - e^{-u}} du = \int_0^\infty \frac{2u^{\sigma-1} e^{(\alpha-1)u}}{1 - e^{-2u}} du \\ &= 2 \int_0^\infty u^{\sigma-1} \sum_{k=0}^\infty e^{-(2k-\alpha+1)u} du = 2 \sum_{k=0}^\infty \int_0^\infty u^{\sigma-1} e^{-(2k-\alpha+1)u} du. \end{aligned}$$

Setting $v = (2k - \alpha + 1)u$ in the above integral, we obtain

$$\begin{aligned} k(\sigma, \alpha) &= 2 \int_0^\infty v^{\sigma-1} e^{-v} dv \sum_{k=0}^\infty \frac{1}{(2k - \alpha + 1)^\sigma} \\ &= 2^{1-\sigma} \Gamma(\sigma) \zeta\left(\sigma, \frac{1-\alpha}{2}\right) \in \mathbf{R}_+, \end{aligned} \tag{2.1}$$

where

$$\Gamma(\eta) := \int_0^\infty v^{\eta-1} e^{-v} dv \quad (\eta > 0)$$

is the gamma function, and

$$\zeta(\sigma, a) := \sum_{k=0}^\infty \frac{1}{(k+a)^\sigma} \quad (a > 0, \sigma > 1)$$

is the Hurwitz zeta function (in particular, $\zeta(\sigma) := \zeta(\sigma, 1) = \sum_{k=1}^\infty \frac{1}{k^\sigma}$ is the Riemann zeta function) (cf. [20]).

For $\alpha = 0$,

$$\begin{aligned} k(\sigma, 0) &= 2^{1-\sigma} \Gamma(\sigma) \zeta\left(\sigma, \frac{1}{2}\right) = 2\Gamma(\sigma) \sum_{k=0}^\infty \frac{1}{(2k+1)^\sigma} \\ &= 2\Gamma(\sigma) \left[\sum_{k=1}^\infty \frac{1}{k^\sigma} - \sum_{k=1}^\infty \frac{1}{(2k)^\sigma} \right] \\ &= 2\Gamma(\sigma) \left(1 - \frac{1}{2^\sigma}\right) \zeta(\sigma); \end{aligned}$$

for $\alpha = -1$, we get

$$k(\sigma, -1) = 2^{1-\sigma} \Gamma(\sigma) \zeta(\sigma, 1) = 2^{1-\sigma} \Gamma(\sigma) \zeta(\sigma).$$

If $p > 1, \frac{1}{p} + \frac{1}{q} = 1, \alpha < 1, \sigma > 1, \sigma_1 \in \mathbf{R}$, for $n \in \mathbf{N} = \{1, 2, \dots\}$, we define the following two expressions:

$$I_1 := \int_1^\infty \left(\int_0^1 e^{\alpha xy} \csc h(xy) x^{\sigma + \frac{1}{pn} - 1} dx \right) y^{\sigma_1 - \frac{1}{qn} - 1} dy, \quad (2.2)$$

$$I_2 := \int_0^1 \left(\int_1^\infty e^{\alpha xy} \csc h(xy) x^{\sigma - \frac{1}{pn} - 1} dx \right) y^{\sigma_1 + \frac{1}{qn} - 1} dy. \quad (2.3)$$

Setting $u = xy$ in (2.2) and (2.3), by Fubini's theorem (cf. [19]), we obtain

$$\begin{aligned} I_1 &= \int_1^\infty y^{(\sigma_1 - \sigma) - \frac{1}{n} - 1} \left(\int_0^y e^{\alpha u} \csc h(u) u^{\sigma + \frac{1}{pn} - 1} du \right) dy \\ &= \int_1^\infty y^{(\sigma_1 - \sigma) - \frac{1}{n} - 1} dy \int_0^1 e^{\alpha u} \csc h(u) u^{\sigma + \frac{1}{pn} - 1} du \\ &\quad + \int_1^\infty \left[\int_u^\infty y^{(\sigma_1 - \sigma) - \frac{1}{n} - 1} dy \right] e^{\alpha u} \csc h(u) u^{\sigma + \frac{1}{pn} - 1} du, \end{aligned} \quad (2.4)$$

$$\begin{aligned} I_2 &= \int_0^1 y^{(\sigma_1 - \sigma) + \frac{1}{n} - 1} \left(\int_y^\infty e^{\alpha u} \csc h(u) u^{\sigma - \frac{1}{pn} - 1} du \right) dy \\ &= \int_0^1 \left[\int_0^u y^{(\sigma_1 - \sigma) + \frac{1}{n} - 1} dy \right] e^{\alpha u} \csc h(u) u^{\sigma - \frac{1}{pn} - 1} du \\ &\quad + \int_0^1 y^{(\sigma_1 - \sigma) + \frac{1}{n} - 1} dy \int_1^\infty e^{\alpha u} \csc h(u) u^{\sigma - \frac{1}{pn} - 1} du. \end{aligned} \quad (2.5)$$

Lemma 2.2. *If $p > 1, \frac{1}{p} + \frac{1}{q} = 1, \alpha < 1, \sigma > 1, \sigma_1 \in \mathbf{R}$, there exists a constant M , such that for any non-negative measurable functions $f(x)$ and $g(y)$ ($x, y \in (0, \infty)$), the following inequality*

$$\begin{aligned} I &:= \int_0^\infty \int_0^\infty e^{\alpha xy} \csc h(xy) f(x) g(y) dx dy \\ &\leq M \left[\int_0^\infty x^{p(1-\sigma) - 1} f^p(x) dx \right]^{\frac{1}{p}} \left[\int_0^\infty y^{q(1-\sigma_1) - 1} g^q(y) dy \right]^{\frac{1}{q}} \end{aligned} \quad (2.6)$$

holds true, then we have $\sigma_1 = \sigma$. For $\sigma_1 = \sigma$, we still have $M \geq k(\sigma, \alpha)$.

Proof. If $\sigma_1 < \sigma$, then for

$$n > \frac{1}{\sigma - \sigma_1} \quad (n \in \mathbf{N}),$$

we set the following two functions

$$f_n(x) := \begin{cases} 0, & 0 < x < 1 \\ x^{\sigma - \frac{1}{pn} - 1}, & x \geq 1 \end{cases}, \quad g_n(y) := \begin{cases} y^{\sigma_1 + \frac{1}{qn} - 1}, & 0 < y \leq 1 \\ 0, & y > 1 \end{cases}.$$

Hence, we obtain

$$\begin{aligned} J_2 & : = \left[\int_0^\infty x^{p(1-\sigma)-1} f_n^p(x) dx \right]^{\frac{1}{p}} \left[\int_0^\infty y^{q(1-\sigma_1)-1} g_n^q(y) dy \right]^{\frac{1}{q}} \\ & = \left(\int_1^\infty x^{-\frac{1}{n}-1} dx \right)^{\frac{1}{p}} \left(\int_0^1 y^{\frac{1}{n}-1} dy \right)^{\frac{1}{q}} = n. \end{aligned}$$

By (2.5) and (2.6), we have

$$\begin{aligned} & \int_0^1 \left[\int_0^u y^{(\sigma_1-\sigma)+\frac{1}{n}-1} dy \right] e^{\alpha u} \csc h(u) u^{\sigma-\frac{1}{pn}-1} du \\ & \leq I_2 = \int_0^\infty \int_0^\infty e^{\alpha xy} \csc h(xy) f_n(x) g_n(y) dx dy \leq M J_2 = Mn. \end{aligned} \quad (2.7)$$

Since

$$(\sigma_1 - \sigma) + \frac{1}{n} < 0,$$

it follows that for any $u \in (0, 1)$,

$$\int_0^u y^{(\sigma_1-\sigma)+\frac{1}{n}-1} dy = \infty.$$

By (2.7), in view of the fact that

$$e^{\alpha u} \csc h(u) u^{\sigma-\frac{1}{pn}-1} > 0, u \in (0, 1),$$

we derive that $\infty < \infty$, which is a contradiction.

If $\sigma_1 > \sigma$, then for

$$n > \frac{1}{\sigma_1 - \sigma} \quad (n \in \mathbf{N}),$$

we set

$$\tilde{f}_n(x) := \begin{cases} x^{\sigma+\frac{1}{pn}-1}, & 0 < x \leq 1 \\ 0, & x > 1 \end{cases}, \quad \tilde{g}_n(y) := \begin{cases} 0, & 0 < y < 1 \\ y^{\sigma_1-\frac{1}{qn}-1}, & y \geq 1 \end{cases}.$$

Hence, we find

$$\begin{aligned} \tilde{J}_2 & : = \left[\int_0^\infty x^{p(1-\sigma)-1} \tilde{f}_n^p(x) dx \right]^{\frac{1}{p}} \left[\int_0^\infty y^{q(1-\sigma_1)-1} \tilde{g}_n^q(y) dy \right]^{\frac{1}{q}} \\ & = \left(\int_0^1 x^{\frac{1}{n}-1} dx \right)^{\frac{1}{p}} \left(\int_1^\infty y^{-\frac{1}{n}-1} dy \right)^{\frac{1}{q}} = n. \end{aligned}$$

By (2.4) and (2.6), we have

$$\begin{aligned} & \int_1^\infty y^{(\sigma_1-\sigma)-\frac{1}{n}-1} dy \int_0^1 e^{\alpha u} \csc h(u) u^{\sigma+\frac{1}{pn}-1} du \\ & \leq I_1 = \int_0^\infty \int_0^\infty e^{\alpha xy} \csc h(xy) \tilde{f}_n(x) \tilde{g}_n(y) dx dy \leq M \tilde{J}_2 = Mn. \end{aligned} \quad (2.8)$$

Since

$$(\sigma_1 - \sigma) - \frac{1}{n} > 0,$$

it follows that

$$\int_1^\infty y^{(\sigma_1 - \sigma) - \frac{1}{n} - 1} dy = \infty.$$

By (2.8), in view of the fact that

$$\int_0^1 e^{\alpha u} \csc h(u) u^{\sigma + \frac{1}{pn} - 1} du > 0,$$

we have $\infty < \infty$, which is a contradiction.

Hence, we conclude that $\sigma_1 = \sigma$.

For $\sigma_1 = \sigma$, we reduce (2.4) and then apply (2.8) as follows:

$$\begin{aligned} \frac{1}{n} I_1 &= \frac{1}{n} \left[\int_1^\infty y^{-\frac{1}{n} - 1} dy \int_0^1 e^{\alpha u} \csc h(u) u^{\sigma + \frac{1}{pn} - 1} du \right. \\ &\quad \left. + \int_1^\infty \left(\int_u^\infty y^{-\frac{1}{n} - 1} dy \right) e^{\alpha u} \csc h(u) u^{\sigma + \frac{1}{pn} - 1} du \right] \\ &= \int_0^1 e^{\alpha u} \csc h(u) u^{\sigma + \frac{1}{pn} - 1} du + \int_1^\infty e^{\alpha u} \csc h(u) u^{\sigma - \frac{1}{qn} - 1} du \\ &\leq \frac{1}{n} M \tilde{J}_2 = M. \end{aligned} \quad (2.9)$$

Since

$$\{e^{\alpha u} \csc h(u) u^{\sigma + \frac{1}{pn} - 1}\}_{n=1}^\infty \quad (\{e^{\alpha u} \csc h(u) u^{\sigma - \frac{1}{qn} - 1}\}_{n=1}^\infty)$$

is nonnegative and increasing in $(0, 1)$ ($(1, \infty)$), by Levi's theorem (cf. [19]), we obtain

$$\begin{aligned} k(\sigma, \alpha) &= \int_0^1 \lim_{n \rightarrow \infty} e^{\alpha u} \csc h(u) u^{\sigma + \frac{1}{pn} - 1} du + \int_1^\infty \lim_{n \rightarrow \infty} e^{\alpha u} \csc h(u) u^{\sigma - \frac{1}{qn} - 1} du \\ &= \lim_{n \rightarrow \infty} \left[\int_0^1 e^{\alpha u} \csc h(u) u^{\sigma + \frac{1}{pn} - 1} du + \int_1^\infty e^{\alpha u} \csc h(u) u^{\sigma - \frac{1}{qn} - 1} du \right] \leq M. \end{aligned} \quad (2.10)$$

The lemma is proved. \square

3. MAIN RESULTS

Theorem 3.1. *If $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $\alpha < 1$, $\sigma > 1$, $\sigma_1 \in \mathbf{R}$, then the following conditions are equivalent:*

(i) *There exists a constant M , such that for any $f(x) \geq 0$, satisfying*

$$0 < \int_0^\infty x^{p(1-\sigma)-1} f^p(x) dx < \infty,$$

we have the following inequality:

$$\begin{aligned} J &: = \left[\int_0^\infty y^{p\sigma_1 - 1} \left(\int_0^\infty e^{\alpha xy} \csc h(xy) f(x) dx \right)^p dy \right]^{\frac{1}{p}} \\ &< M \left[\int_0^\infty x^{p(1-\sigma)-1} f^p(x) dx \right]^{\frac{1}{p}}; \end{aligned} \quad (3.1)$$

(ii) There exists a constant M , such that for any $f(x), g(y) \geq 0$, satisfying

$$0 < \int_0^\infty x^{p(1-\sigma)-1} f^p(x) dx < \infty,$$

and

$$0 < \int_0^\infty y^{q(1-\sigma_1)-1} g^q(y) dy < \infty,$$

we have the following Hilbert-type integral inequality:

$$\begin{aligned} I &= \int_0^\infty \int_0^\infty e^{\alpha xy} \operatorname{csc} h(xy) f(x) g(y) dx dy \\ &< M \left[\int_0^\infty x^{p(1-\sigma)-1} f^p(x) dx \right]^{\frac{1}{p}} \left[\int_0^\infty y^{q(1-\sigma_1)-1} g^q(y) dy \right]^{\frac{1}{q}}; \end{aligned} \quad (3.2)$$

(iii) $\sigma_1 = \sigma$.

If Condition (iii) follows, then $M \geq k(\sigma, \alpha)$ and the constant factor

$$M = k(\sigma, \alpha) = 2^{1-\sigma} \Gamma(\sigma) \zeta\left(\sigma, \frac{1-\alpha}{2}\right)$$

in (3.1) and (3.2) is the best possible.

Proof. (i) \implies (ii). By Hölder's inequality (cf. [21]), we have

$$\begin{aligned} I &= \int_0^\infty \left(y^{\sigma_1 - \frac{1}{p}} \int_0^\infty e^{\alpha xy} \operatorname{csc} h(xy) f(x) dx \right) \left(y^{\frac{1}{p} - \sigma_1} g(y) \right) dy \\ &\leq J \left[\int_0^\infty y^{q(1-\sigma_1)-1} g^q(y) dy \right]^{\frac{1}{q}}. \end{aligned} \quad (3.3)$$

Then by (3.1), we get (3.2).

(ii) \implies (iii). By Lemma 2.2, we have $\sigma_1 = \sigma$.

(iii) \implies (i). Setting $u = xy$, we obtain the following weight function:

$$\begin{aligned} \omega(\sigma, y) &: = y^\sigma \int_0^\infty e^{\alpha xy} \operatorname{csc} h(xy) x^{\sigma-1} dx \\ &= \int_0^\infty e^{\alpha u} \operatorname{csc} h(u) u^{\sigma-1} du = k(\sigma, \alpha) \quad (y > 0). \end{aligned} \quad (3.4)$$

By Hölder's inequality with weight and (3.4), we have

$$\begin{aligned}
& \left(\int_0^\infty e^{\alpha xy} \csc h(xy) f(x) dx \right)^p \\
&= \left\{ \int_0^\infty e^{\alpha xy} \csc h(xy) \left[\frac{y^{(\sigma-1)/p}}{x^{(\sigma-1)/q}} f(x) \right] \left[\frac{x^{(\sigma-1)/q}}{y^{(\sigma-1)/p}} \right] dx \right\}^p \\
&\leq \int_0^\infty e^{\alpha xy} \csc h(xy) \frac{y^{\sigma-1}}{x^{(\sigma-1)p/q}} f^p(x) dx \\
&\quad \times \left[\int_0^\infty e^{\alpha xy} \csc h(xy) \frac{x^{\sigma-1}}{y^{(\sigma-1)q/p}} dx \right]^{p/q} \\
&= [\omega(\sigma, y) y^{q(1-\sigma)-1}]^{p-1} \int_0^\infty e^{\alpha xy} \csc h(xy) \frac{y^{\sigma-1}}{x^{(\sigma-1)p/q}} f^p(x) dx \\
&= (k(\sigma, \alpha))^{p-1} y^{-p\sigma+1} \int_0^\infty e^{\alpha xy} \csc h(xy) \frac{y^{\sigma-1}}{x^{(\sigma-1)p/q}} f^p(x) dx \tag{3.5}
\end{aligned}$$

If (3.5) takes the form of an equality for some $y \in (0, \infty)$, then (cf. [21]) there exist constants A and B , such that they are not all zero, and

$$A \frac{y^{\sigma-1}}{x^{(\sigma-1)p/q}} f^p(x) = B \frac{x^{\sigma-1}}{y^{(\sigma-1)q/p}} \quad a.e. \text{ in } \mathbf{R}_+.$$

We assume that $A \neq 0$ (otherwise $B = A = 0$). Then it follows that

$$x^{p(1-\sigma)-1} f^p(x) = y^{q(1-\sigma)} \frac{B}{Ax} \quad a.e. \text{ in } \mathbf{R}_+,$$

which contradicts the fact that

$$0 < \int_0^\infty x^{p(1-\sigma)-1} f^p(x) dx < \infty.$$

Hence, (3.5) takes the form of strict inequality.

For $\sigma_1 = \sigma$, by Fubini's theorem, we have

$$\begin{aligned}
J &< (k(\sigma, \alpha))^{\frac{1}{q}} \left[\int_0^\infty \int_0^\infty e^{\alpha xy} \csc h(xy) \frac{y^{\sigma-1}}{x^{(\sigma-1)p/q}} f^p(x) dx dy \right]^{\frac{1}{p}} \\
&= (k(\sigma, \alpha))^{\frac{1}{q}} \left\{ \int_0^\infty \left[\int_0^\infty e^{\alpha xy} \csc h(xy) \frac{y^{\sigma-1}}{x^{(\sigma-1)(p-1)}} dy \right] f^p(x) dx \right\}^{\frac{1}{p}} \\
&= (k(\sigma, \alpha))^{\frac{1}{q}} \left[\int_0^\infty \omega(\sigma, x) x^{p(1-\sigma)-1} f^p(x) dx \right]^{\frac{1}{p}} \\
&= k(\sigma, \alpha) \left[\int_0^\infty x^{p(1-\sigma)-1} f^p(x) dx \right]^{\frac{1}{p}}.
\end{aligned}$$

Setting $M \geq k(\sigma, \alpha)$, inequality (3.1) follows.

Therefore, the conditions (i), (ii) and (iii) are equivalent.

In the case when condition (iii) follows, if there exists a constant $M < k(\sigma, \alpha)$, such that (3.2) is valid, then by Lemma 2.2 we have $M \geq k(\sigma, \alpha)$. From this contradiction it follows that the constant factor $M = k(\sigma, \alpha)$ in (3.2) is the best

possible. The constant factor $M = k(\sigma, \alpha)$ in (3.1) is still the best possible. Otherwise, by (3.3) (for $\sigma_1 = \sigma$), we can conclude that the constant factor $M = k(\sigma, \alpha)$ in (3.2) is not the best possible. \square

Setting $y = \frac{1}{Y}$, $G(Y) = \frac{1}{Y^2}g(\frac{1}{Y})$ in Theorem 3.1, then replacing Y ($G(Y)$) by y ($g(y)$), we obtain the following corollary

Corollary 3.2. *If $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $\alpha < 1$, $\sigma > 1$, $\sigma_1 \in \mathbf{R}$, then the following conditions are equivalent:*

(i) *There exists a constant M , such that for any $f(x) \geq 0$, satisfying*

$$0 < \int_0^\infty x^{p(1-\sigma)-1} f^p(x) dx < \infty,$$

we have the following inequality:

$$\left\{ \int_0^\infty y^{-p\sigma_1-1} \left[\int_0^\infty e^{\alpha x/y} \operatorname{csc} h(x/y) f(x) dx \right]^p dy \right\}^{\frac{1}{p}} < M \left[\int_0^\infty x^{p(1-\sigma)-1} f^p(x) dx \right]^{\frac{1}{p}}; \quad (3.6)$$

(ii) *There exists a constant M , such that for any $f(x), g(y) \geq 0$, satisfying*

$$0 < \int_0^\infty x^{p(1-\sigma)-1} f^p(x) dx < \infty,$$

and

$$0 < \int_0^\infty y^{q(1+\sigma_1)-1} g^q(y) dy < \infty,$$

we have the following Hilbert-type integral inequality with the homogeneous kernel:

$$\int_0^\infty \int_0^\infty e^{\alpha x/y} \operatorname{csc} h(x/y) f(x) g(y) dx dy < M \left[\int_0^\infty x^{p(1-\sigma)-1} f^p(x) dx \right]^{\frac{1}{p}} \left[\int_0^\infty y^{q(1+\sigma_1)-1} g^q(y) dy \right]^{\frac{1}{q}}; \quad (3.7)$$

(iii) $\sigma_1 = \sigma$.

In the case when the condition (iii) holds, we get $M \geq k(\sigma, \alpha)$, and the constant $M = k(\sigma, \alpha)$ in (3.6) and (3.7) is the best possible.

Remark 3.3. On the other hand, setting $y = \frac{1}{Y}$, $G(Y) = \frac{1}{Y^2}g(\frac{1}{Y})$, in Corollary 3.2, then replacing Y ($G(Y)$) by y ($g(y)$), we obtain Theorem 3.1. Hence, Theorem 3.1 and Corollary 3.2 are equivalent.

4. OPERATOR EXPRESSIONS

We set the following functions:

$$\varphi(x) := x^{p(1-\sigma)-1}, \quad \psi(y) := y^{q(1-\sigma)-1}, \quad \phi(y) := y^{q(1+\sigma)-1},$$

wherefrom,

$$\psi^{1-p}(y) = y^{p\sigma-1}, \quad \phi^{1-p}(y) = y^{-p\sigma-1} \quad (x, y \in \mathbf{R}_+),$$

and define the following real normed linear spaces:

$$L_{p,\varphi}(\mathbf{R}_+) := \left\{ f : \|f\|_{p,\varphi} := \left(\int_0^\infty \varphi(x) |f(x)|^p dx \right)^{\frac{1}{p}} < \infty \right\},$$

wherefrom,

$$L_{q,\psi}(\mathbf{R}_+) = \left\{ g : \|g\|_{q,\psi} := \left(\int_0^\infty \psi(y) |g(y)|^q dy \right)^{\frac{1}{q}} < \infty \right\},$$

$$L_{q,\phi}(\mathbf{R}_+) = \left\{ g : \|g\|_{q,\phi} := \left(\int_0^\infty \phi(y) |g(y)|^q dy \right)^{\frac{1}{q}} < \infty \right\},$$

$$L_{p,\psi^{1-p}}(\mathbf{R}_+) = \left\{ h : \|h\|_{p,\psi^{1-p}} = \left(\int_0^\infty \psi^{1-p}(y) |h(y)|^p dy \right)^{\frac{1}{p}} < \infty \right\},$$

$$L_{q,\phi^{1-p}}(\mathbf{R}_+) = \left\{ h : \|h\|_{p,\phi^{1-p}} = \left(\int_0^\infty \phi^{1-p}(y) |h(y)|^p dy \right)^{\frac{1}{p}} < \infty \right\}.$$

(a) In view of Theorem 3.1 (when $\sigma_1 = \sigma$), for $f \in L_{p,\varphi}(\mathbf{R}_+)$, setting

$$h_1(y) := \int_0^\infty e^{\alpha xy} \csc h(xy) f(x) dx \quad (y \in \mathbf{R}_+),$$

by (3.1), we have

$$\|h_1\|_{p,\psi^{1-p}} = \left[\int_0^\infty \psi^{1-p}(y) h_1^p(y) dy \right]^{\frac{1}{p}} < M \|f\|_{p,\varphi} < \infty. \quad (4.1)$$

Definition 4.1. Define a Hilbert-type integral operator with the non-homogeneous kernel $T^{(1)} : L_{p,\varphi}(\mathbf{R}_+) \rightarrow L_{p,\psi^{1-p}}(\mathbf{R}_+)$ as follows: For any $f \in L_{p,\varphi}(\mathbf{R}_+)$, there exists a unique representation $T^{(1)}f = h_1 \in L_{p,\psi^{1-p}}(\mathbf{R}_+)$, satisfying $T^{(1)}f(y) = h_1(y)$, for any $y \in \mathbf{R}_+$.

In view of (4.1), it follows that

$$\|T^{(1)}f\|_{p,\psi^{1-p}} = \|h_1\|_{p,\psi^{1-p}} \leq M \|f\|_{p,\varphi},$$

and then the operator $T^{(1)}$ is bounded satisfying

$$\|T^{(1)}\| = \sup_{f(\neq\theta) \in L_{p,\varphi}(\mathbf{R}_+)} \frac{\|T^{(1)}f\|_{p,\psi^{1-p}}}{\|f\|_{p,\varphi}} \leq M.$$

If we define the formal inner product of $T^{(1)}f$ and g by:

$$(T^{(1)}f, g) := \int_0^\infty \left(\int_0^\infty e^{\alpha xy} \csc h(xy) f(x) dx \right) g(y) dy,$$

then we can rewrite Theorem 3.1 as follows:

Theorem 4.2. *If $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $\alpha < 1$, $\sigma > 1$, then the following conditions are equivalent:*

(i) There exists a constant M , such that for any

$$f(x) \geq 0, f \in L_{p,\varphi}(\mathbf{R}_+), \|f\|_{p,\varphi} > 0,$$

the following inequality holds true:

$$\|T^{(1)}f\|_{p,\psi^{1-p}} < M\|f\|_{p,\varphi}; \quad (4.2)$$

(ii) There exists a constant M , such that for any

$$f(x), g(y) \geq 0, f \in L_{p,\varphi}(\mathbf{R}_+), g \in L_{q,\psi}(\mathbf{R}_+), \|f\|_{p,\varphi}, \|g\|_{q,\psi} > 0,$$

the following inequality holds true:

$$(T^{(1)}f, g) < M\|f\|_{p,\varphi}\|g\|_{q,\psi}. \quad (4.3)$$

We still have $\|T^{(1)}\| = k(\sigma, \alpha) \leq M$.

(b) In view of Corollary 3.2 (when $\sigma_1 = \sigma$), for $f \in L_{p,\varphi}(\mathbf{R}_+)$, setting

$$h_2(y) := \int_0^\infty e^{\alpha x/y} \csc h(x/y) f(x) dx \quad (y \in \mathbf{R}_+),$$

by (3.6), we have

$$\|h_2\|_{p,\phi^{1-p}} = \left[\int_0^\infty \phi^{1-p}(y) h_2^p(y) dy \right]^{\frac{1}{p}} < M\|f\|_{p,\varphi} < \infty. \quad (4.4)$$

Definition 4.3. Define a Hilbert-type integral operator with the homogeneous kernel $T^{(2)} : L_{p,\varphi}(\mathbf{R}_+) \rightarrow L_{p,\phi^{1-p}}(\mathbf{R}_+)$ as follows: For any $f \in L_{p,\varphi}(\mathbf{R}_+)$, there exists a unique representation $T^{(2)}f = h_2 \in L_{p,\phi^{1-p}}(\mathbf{R}_+)$, satisfying $T^{(2)}f(y) = h_2(y)$, for any $y \in \mathbf{R}_+$.

In view of (4.4), it follows that

$$\|T^{(2)}f\|_{p,\phi^{1-p}} = \|h_2\|_{p,\phi^{1-p}} \leq M\|f\|_{p,\varphi},$$

and thus the operator $T^{(2)}$ is bounded satisfying

$$\|T^{(2)}\| = \sup_{f(\neq\theta) \in L_{p,\varphi}(\mathbf{R}_+)} \frac{\|T^{(2)}f\|_{p,\phi^{1-p}}}{\|f\|_{p,\varphi}} \leq M.$$

If we define the formal inner product of $T^{(2)}f$ and g by:

$$(T^{(2)}f, g) := \int_0^\infty \left(\int_0^\infty e^{\alpha x/y} \csc h(x/y) f(x) dx \right) g(y) dy,$$

then we can rewrite Corollary 3.2 as follows:

Corollary 4.4. *If $p > 1, \frac{1}{p} + \frac{1}{q} = 1, \alpha < 1, \sigma > 1$, then the following conditions are equivalent:*

(i) *There exists a constant M , such that for any*

$$f(x) \geq 0, f \in L_{p,\varphi}(\mathbf{R}_+), \|f\|_{p,\varphi} > 0,$$

the following inequality holds true:

$$\|T^{(2)}f\|_{p,\phi^{1-p}} < M\|f\|_{p,\varphi}; \quad (4.5)$$

(ii) *There exists a constant M , such that for any*

$$f(x), g(y) \geq 0, f \in L_{p,\varphi}(\mathbf{R}_+), g \in L_{q,\phi}(\mathbf{R}_+), \|f\|_{p,\varphi}, \|g\|_{q,\phi} > 0,$$

the following inequality holds true:

$$(T^{(2)}f, g) < M\|f\|_{p,\varphi}\|g\|_{q,\phi}. \quad (4.6)$$

We still have $\|T^{(2)}\| = k(\sigma, \alpha) \leq M$.

Remark 4.5. Theorem 4.2 and Corollary 4.4 are equivalent.

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