A Linear Functional Equation of Third Order associated to the Fibonacci Numbers

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Abstract

Given a vector space \( X \), we investigate the solutions \( f: \mathbb{R} \to X \) of the linear functional equation of third order

\[
f(x) = pf(x - 1) + qf(x - 2) + rf(x - 3),
\]

which is strongly associated to a well known identity for the Fibonacci numbers. Moreover, we prove the Hyers-Ulam stability of that equation.

1 Introduction

In 1940, S.M. Ulam [33] gave a wide ranging talk before the mathematics club of the University of Wisconsin in which he discussed a number of important unsolved problems. Among those was the question concerning the stability of group homomorphisms:

Let \( G_1 \) be a group and let \( G_2 \) be a metric group with the metric \( d(\cdot, \cdot) \). Given \( \varepsilon > 0 \), does there exist a \( \delta > 0 \) such that if a function \( h: G_1 \to G_2 \) satisfies the inequality \( d(h(xy), h(x)h(y)) < \delta \) for all \( x, y \in G_1 \), then there exists a homomorphism \( H: G_1 \to G_2 \) with \( d(h(x), H(x)) < \varepsilon \) for all \( x \in G_1 \) ?

The case of approximately additive functions was solved by D.H. Hyers [12] under the assumption that \( G_1 \) and \( G_2 \) are Banach spaces. Indeed, he proved the following theorem

Theorem 1.1 Let \( f: G_1 \to G_2 \) be a function between Banach spaces such that

\[
\|f(x + y) - f(x) - f(y)\| \leq \varepsilon
\]

for some \( \varepsilon > 0 \) and for all \( x, y \in G_1 \). Then the limit

\[
A(x) := \lim_{n \to \infty} 2^{-n} f(2^n x)
\]

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exists for each \( x \in G_1 \), and \( A : G_1 \to G_2 \) is the unique additive function such that
\[
\| f(x) - A(x) \| \leq \varepsilon
\]
for any \( x \in G_1 \). Moreover, if \( f(tx) \) is continuous in \( t \) for each fixed \( x \in G_1 \), then the function \( A \) is linear.

Hyers proved that each solution of the inequality \( \| f(x+y) - f(x) - f(y) \| \leq \varepsilon \) can be approximated by an exact solution, say an additive function. In this case, the Cauchy additive functional equation, \( f(x+y) = f(x) + f(y) \), is said to have the Hyers-Ulam stability.

Since then, the stability problems of a large variety of functional equations have been extensively investigated by several mathematicians (cf. [8, 13, 14, 16, 17, 22, 24, 26]). For further discussion and references one is referred to [5, 6, 7, 23, 30].

In this paper, as usual, \( \mathbb{C}, \mathbb{R}, \mathbb{Z} \) and \( \mathbb{N} \) stand for the sets of complex numbers, real numbers, integers, and positive integers, respectively. For a nonempty subset \( S \) of a vector space, let \( \xi : S \to S \) be a function. Moreover, \( \xi^0(x) = x, \xi^{n+1}(x) = \xi(\xi^n(x)) \) and (only for bijective \( \xi \)) \( \xi^{-n-1}(x) = \xi^{-1}(\xi^{-n}(x)) \) for \( x \in S \) and \( n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\} \).

S.-M. Jung has proved in [14] (see also [15]) some results on solutions and Hyers-Ulam stability of the functional equation
\[
f(x) = pf(\xi(x)) - qf(\xi^2(x)),
\]
in the case where \( S = \mathbb{R} \) and \( \xi(x) = x - 1 \) for \( x \in \mathbb{R} \).

If \( S := \mathbb{N}_0 \) and \( p, q \in \mathbb{Z} \), then solutions \( x : \mathbb{N}_0 \to \mathbb{Z} \) of the difference equation \( f(x) = pf(x-1) - qf(x-2) \) are called the Lucas sequences (see, e.g., [29]). In some special cases they are called with specific names, for example, the Fibonacci numbers \((p = 1, q = -1), x(0) = 0, x(1) = 1\), the Lucas numbers \((p = 1, q = -1), x(0) = 2, x(1) = 1\), the Pell numbers \((p = 2, q = -1), x(0) = 0, x(1) = 1\), the Pell-Lucas (or companion Lucas) numbers \((p = 2, q = -1), x(0) = 2, x(1) = 2\), and the Jacobsthal numbers \((p = 1, q = -2), x(0) = 0, x(1) = 1\).

For some information and further references concerning the functional equations in a single variable, we refer to [1, 20, 21]. Let us mention yet that the problem of Hyers-Ulam stability of functional equations is connected to the notions of controlled chaos (see [31]) and shadowing (see [11, 25, 27]).

We remark that if \( \xi : S \to S \) is bijective, then Eq. (1.1) can be written in the following equivalent form
\[
f(\eta^2(x)) = pf(\eta(x)) - qf(x),
\]
where \( \eta := \xi^{-1} \).

In view of the last remark, the following Hyers-Ulam stability result concerning equation (1.1) can be derived from [4, Theorem 2] (see also [32]).

**Theorem 1.2** Let \( p, q \in \mathbb{R} \) be given with \( q \neq 0 \) and let \( S \) be a nonempty subset of a vector space. Assume that \( a_1, a_2 \) are the complex roots of the quadratic equation \( x^2 - px + q = 0 \) with \( |a_i| \neq 1 \) for \( i \in \{1, 2\} \). Moreover, assume that \( X \) is either a real vector space if \( p^2 - 4q > 0 \) or a complex vector space if \( p^2 - 4q < 0 \). Let \( \xi : S \to S \) be bijective. If a function \( f : S \to X \) satisfies the inequality
\[
\| f(x) - pf(\xi(x)) + qf(\xi^2(x)) \| \leq \varepsilon
\]
(1.2)
for all \( x \in S \) and for some \( \varepsilon \geq 0 \), then there exists a unique solution \( F : S \rightarrow X \) of (1.1) with
\[
\| f(x) - F(x) \| \leq \frac{\varepsilon}{(|a_1| - 1)(|a_2| - 1)}
\]
for all \( x \in S \).

In [2, Theorem 1.4], the method presented in [14] was modified so as to prove a theorem which is a complement of Theorem 1.2. Note that, for bijective \( \xi \), the following theorem improves the estimation (1.3) in some cases (e.g., \( a_1 = 3/2, a_2 = -3/2 \), or \( a_1 = 1/2, a_2 = -1/2 \)). However, in some other situations (e.g., \( a_1 = 3, a_2 = -3 \)), the estimation (1.3) is better than (1.4). The following theorem also complements Theorem 1.2, because \( \xi \) can be quite arbitrary in the case of (a).

**Theorem 1.3** Given \( p, q \in \mathbb{R} \) with \( q \neq 0 \), assume that the distinct complex roots \( a_1, a_2 \) of the quadratic equation \( x^2 - px + q = 0 \) satisfy one of the following two conditions:

(a) \( |a_i| < 1 \) for \( i \in \{1, 2\} \);

(b) \( |a_i| \neq 1 \) for \( i \in \{1, 2\} \) and \( \xi : S \rightarrow S \) is bijective.

Moreover, assume that \( X \) is either a real vector space if \( p^2 - 4q > 0 \) or a complex vector space if \( p^2 - 4q < 0 \). If a function \( f : S \rightarrow X \) satisfies the inequality (1.2), then there exists a solution \( F : S \rightarrow X \) of Eq. (1.1) such that
\[
\| f(x) - F(x) \| \leq \frac{\varepsilon}{|a_1 - a_2|} \left( \frac{|a_1|}{|a_1| - 1} + \frac{|a_2|}{|a_2| - 1} \right)
\]
for all \( x \in S \). Moreover, if the condition (b) is true, then the \( F \) is the unique solution of Eq. (1.1) satisfying (1.4).

In this paper, we investigate the solutions of the functional equation
\[
f(x) = pf(x-1) + qf(x-2) + rf(x-3),
\]
where \( p, q, r \) are real constants. Moreover, we also prove the Hyers-Ulam stability of that equation. The equation (1.5) is a kind of linear functional equations of third order because it is of the form
\[
f(x) = a_1(x)f(\xi(x)) + a_2(x)f(\xi^2(x)) + a_3(x)f(\xi^3(x))
\]
for the case of \( a_1(x) = p, a_2(x) = q, a_3(x) = r, \) and \( \xi(x) = x - 1 \).

## 2 General solution

In the following theorem, we apply [2, Theorem 1.1] for the investigation of general solutions of the functional equation (1.5).

**Theorem 2.1** Let \( p, q, r \) be real constants such that the cubic equation
\[
x^3 + px^2 - qx + r = 0
\]
has the following properties:
(i) \( \alpha_1 \) and \( \alpha_2 \) are two distinct nonzero roots of the cubic equation (2.1);

(ii) it holds true that either \((\alpha_i + p)^2 + 4r/\alpha_i > 0\) for \(i \in \{1, 2\}\) or \((\alpha_i + p)^2 + 4r/\alpha_i < 0\) for \(i \in \{1, 2\}\).

Let \( X \) be either a real vector space if \((\alpha_i + p)^2 + 4r/\alpha_i > 0\) for \(i \in \{1, 2\}\) or a complex vector space if \((\alpha_i + p)^2 + 4r/\alpha_i < 0\) for \(i \in \{1, 2\}\). Then, a function \( f : \mathbb{R} \to X \) is a solution of the functional equation (1.5) if and only if there exist functions \( h_1, h_2 : [-1, 1) \to X \) such that

\[
 f(x) = \frac{\alpha_1}{\alpha_1 - \alpha_2} V_{[x]+1} h_2(x - [x]) + \frac{\alpha_1 r}{\alpha_2 (\alpha_1 - \alpha_2)} V_{[x]} h_2(x - [x] - 1) \\
- \frac{\alpha_2}{\alpha_1 - \alpha_2} U_{[x]+1} h_1(x - [x]) - \frac{\alpha_2 r}{\alpha_1 (\alpha_1 - \alpha_2)} U_{[x]} h_1(x - [x] - 1),
\]

where \([x]\) denotes the largest integer not exceeding \(x\), and \( U_n, V_n \) are defined in (2.5) and (2.13).

**Proof.** Assume that \( f : \mathbb{R} \to X \) is a solution of Eq. (1.5). If we define an auxiliary function \( g_1 : \mathbb{R} \to X \) by

\[
 g_1(x) := f(x) + \alpha_1 f(x - 1),
\]

then it follows from (1.5) that \( g_1 \) satisfies

\[
 g_1(x) = (\alpha_1 + p) g_1(x - 1) + \frac{r}{\alpha_1} g_1(x - 2)
\]

for any \(x \in \mathbb{R}\). According to [2, Theorem 1.1], there exists a function \( h_1 : [-1, 1) \to X \) such that

\[
 g_1(x) = f(x) + \alpha_1 f(x - 1) = U_{[x]+1} h_1(x - [x]) + \frac{r}{\alpha_1} U_{[x]} h_1(x - [x] - 1)
\]

for all \(x \in \mathbb{R}\), where

\[
 U_n = \frac{a^n - b^n}{a - b} \quad (n \in \mathbb{Z})
\]

and \(a, b\) are the distinct roots of the quadratic equation

\[
 x^2 - (\alpha_1 + p)x - r/\alpha_1 = 0,
\]

i.e.,

\[
 a = \frac{\alpha_1 + p}{2} + \sqrt{\left(\frac{\alpha_1 + p}{2}\right)^2 + \frac{r}{\alpha_1}}, \quad b = \frac{\alpha_1 + p}{2} - \sqrt{\left(\frac{\alpha_1 + p}{2}\right)^2 + \frac{r}{\alpha_1}}.
\]

Since \(a\) is a root of the quadratic equation (2.6), we have

\[
 a^2 = (\alpha_1 + p)a + \frac{r}{\alpha_1}.
\]
We multiply both sides of (2.8) with \(a\) and make use of (2.8) and (i) to get
\[
a^3 = pa^2 + \alpha_a + \frac{r}{\alpha_1}a
\]
\[
= pa^2 + \alpha_a \left( (\alpha_1 + p)a + \frac{r}{\alpha_1} \right) + \frac{r}{\alpha_1}a
\]
\[
= pa^2 + \frac{\alpha}{\alpha_1} (\alpha_1^2 + p\alpha_1 + r) + r
\]
\[
= pa^2 + qa + r.
\] (2.9)

Similarly, we also obtain
\[
b^3 = pb^2 + qb + r.
\] (2.10)

Using (2.5), (2.9), and (2.10), we have
\[
pU_{n-1} + qU_{n-2} + rU_{n-3} = \frac{(pa^2 + qa + r)a^{n-3} - (pb^2 + qb + r)b^{n-3}}{a - b} = \frac{a^n - b^n}{a - b} = U_n
\] (2.11)

for all \(n \in \mathbb{Z}\).

If we define an auxiliary function \(g_2 : \mathbb{R} \to X\) by
\[
g_2(x) := f(x) + \alpha_2 f(x - 1),
\]
then it follows from (1.5) that \(g_2\) satisfies
\[
g_2(x) = (\alpha_2 + p)g_2(x - 1) + \frac{r}{\alpha_2}g_2(x - 2)
\]
for any \(x \in \mathbb{R}\). According to [2, Theorem 1.1], there exists a function \(h_2 : [-1, 1) \to X\) such that
\[
g_2(x) = f(x) + \alpha_2 f(x - 1) = V_{[x]+1}h_2(x - [x]) + \frac{r}{\alpha_2}V_{[x]}h_2(x - [x] - 1)
\] (2.12)
for all \(x \in \mathbb{R}\), where
\[
V_n = \frac{c^n - d^n}{c - d} \quad (n \in \mathbb{Z})
\] (2.13)
and \(c, d\) are the distinct roots of the quadratic equation
\[
x^2 - (\alpha_2 + p)x - r/\alpha_2 = 0,
\]
i.e.,
\[
c = \frac{\alpha_2 + p}{2} + \sqrt{\left( \frac{\alpha_2 + p}{2} \right)^2 + \frac{r}{\alpha_2}}, \quad d = \frac{\alpha_2 + p}{2} - \sqrt{\left( \frac{\alpha_2 + p}{2} \right)^2 + \frac{r}{\alpha_2}}.
\] (2.14)

As in the first part, we verify that
\[
V_n = pV_{n-1} + qV_{n-2} + rV_{n-3}
\] (2.15)
for all $n \in \mathbb{Z}$.

We now multiply (2.4) with $\alpha_2$ and (2.12) with $\alpha_1$, we subtract the former from the latter, and we then divide the resulting equation by $(\alpha_1 - \alpha_2)$ to get (2.2).

We assume that a function $f : \mathbb{R} \to X$ is given by (2.2), where $h_1, h_2 : [-1, 1) \to X$ are arbitrarily given functions, and $U_n, V_n$ are given by (2.5) and (2.13), respectively. Then, by (2.2), (2.11), and (2.15), we have

$$p f(x - 1) + q f(x - 2) + r f(x - 3)$$

$$= \frac{\alpha_1}{\alpha_1 - \alpha_2} (p V[x] + q V[x] - 1 + r V[x] - 2) h_2(x - [x])$$
$$+ \frac{\alpha_1 r}{\alpha_2} (p V[x] - 1 + q V[x] - 2 + r V[x] - 3) h_2(x - [x] - 1)$$
$$- \frac{\alpha_2}{\alpha_1 - \alpha_2} \left( p U[x] + q U[x] - 1 + r U'[x] - 2 \right) h_1(x - [x])$$
$$- \frac{\alpha_2}{\alpha_1 - \alpha_2} \left( p U'[x] + q U'[x] - 1 + r U''[x] - 2 \right) h_1(x - [x] - 1)$$
$$+ \frac{\alpha_1}{\alpha_1 - \alpha_2} V'[x] + 1 h_2(x - [x])$$
$$+ \frac{\alpha_1}{\alpha_1 - \alpha_2} V'[x] + 1 h_2(x - [x] - 1)$$
$$= f(x)$$

for all $x \in \mathbb{R}$, which implies that $f(x)$ is a solution of (1.5). 

According to [19, p. 92], the Fibonacci numbers $F_n$ satisfy the identity

$$F_n^2 = 2F_{n-1}^2 + 2F_{n-2}^2 - F_{n-3}^2$$

(2.16)

for all integers $n > 3$. We can easily notice that the linear equation of third order

$$f(x) = 2 f(x - 1) + 2 f(x - 2) - f(x - 3)$$

(2.17)

is strongly related to the identity (2.16).

**Corollary 2.2** Let $X$ be a real vector space. A function $f : \mathbb{R} \to X$ is a solution of the functional equation (2.17) if and only if there exist functions $h_1, h_2 : [-1, 1) \to X$ such that

$$f(x) = \frac{5 + 3 \sqrt{5}}{10} U'[x] + 1 h_1(x - [x]) + \frac{15 + 7 \sqrt{5}}{10} U'[x] h_1(x - [x] - 1)$$
$$+ \frac{5 - 3 \sqrt{5}}{10} V'[x] + 1 h_2(x - [x]) + \frac{15 - 7 \sqrt{5}}{10} V'[x] h_2(x - [x] - 1),$$

where $U'_n$ and $V'_n$ are defined in (2.18).

**Proof.** If we set $p = 2$, $q = 2$, and $r = -1$ in (2.1), then the cubic equation

$$x^3 + 2x^2 - 2x - 1 = 0$$

has three distinct nonzero roots including

$$\alpha_1 = \frac{-3 + \sqrt{5}}{2} \text{ and } \alpha_2 = \frac{-3 - \sqrt{5}}{2}.$$
Moreover, it holds that \((\alpha_1 + p)^2 + 4r/\alpha_1 > 0\) and \((\alpha_2 + p)^2 + 4r/\alpha_2 > 0\). By (2.5), (2.7), (2.13), and (2.14), we have

\[ U_n' = \frac{a^n - b^n}{a - b} \quad \text{and} \quad V_n' = \frac{c^n - d^n}{c - d}, \tag{2.18} \]

where we make use of (2.7) and (2.14) to calculate

\[ a = \frac{3 + \sqrt{5}}{2}, \quad b = -1, \quad c = \frac{3 - \sqrt{5}}{2}, \quad d = -1. \]

Finally, in view of Theorem 2.1, we conclude that the assertion of our corollary is true. \(\square\)

**Corollary 2.3** If a function \(f : \mathbb{R} \to \mathbb{R}\) is a solution of the functional equation (2.17), then there exist real constants \(\mu_1, \mu_2, \nu_1, \text{ and } \nu_2\) such that

\[ f(n) = \frac{5 + 3\sqrt{5}}{10} \mu_1 U_{n+1}' + \frac{15 + 7\sqrt{5}}{10} \mu_2 U_n' + \frac{5 - 3\sqrt{5}}{10} \nu_1 V_{n+1}' + \frac{15 - 7\sqrt{5}}{10} \nu_2 V_n' \]

for all \(n \in \mathbb{Z}\), where \(U_n'\) and \(V_n'\) are defined in (2.18).

### 3 Hyers-Ulam stability

**Theorem 3.1** Let \(p, q, r\) be real constants with \(r \neq 0\), \(\alpha\) be a nonzero root of the cubic equation (2.1), and let \(a, b\) be the roots of the quadratic equation \(x^2 - (\alpha + p)x - r/\alpha = 0\) with \(|a| > 1\) and \(0 < |b| < 1\). Assume that \((\alpha + p)^2 + 4r/\alpha \neq 0\). Let \(X\) be either a real Banach space if \((\alpha + p)^2 + 4r/\alpha > 0\) or a complex Banach space if \((\alpha + p)^2 + 4r/\alpha < 0\). If a function \(f : \mathbb{R} \to X\) satisfies the inequality

\[ \|f(x) - pf(x - 1) - qf(x - 2) - rf(x - 3)\| \leq \varepsilon \]  \(\text{for all } x \in \mathbb{R} \text{ and for some } \varepsilon \geq 0,\) then there exists a solution \(G : \mathbb{R} \to X\) of Eq. (1.5) such that

\[ \|f(x) + \alpha f(x - 1) - G(x)\| \leq \frac{|a| - |b|}{|a - b| (|a| - 1)(1 - |b|)} \varepsilon \]  \(\text{for all } x \in \mathbb{R}.\)

**Proof.** If we define an auxiliary function \(g : \mathbb{R} \to X\) by

\[ g(x) := f(x) + \alpha f(x - 1), \]

then, as we did in (2.3), it follows from (3.1) that \(g\) satisfies the inequality

\[ \left\| g(x) - (\alpha + p)g(x - 1) - \frac{r}{\alpha} g(x - 2) \right\| \leq \varepsilon \]

or

\[ \left\| g(x) - ag(x - 1) - b[g(x - 1) - ag(x - 2)] \right\| \leq \varepsilon \]
for any \( x \in \mathbb{R} \).

If we replace \( x \) with \( x - k \) in the last inequality, then we have
\[
\| g(x - k) - ag(x - k - 1) - b[g(x - k - 1) - ag(x - k - 2)] \| \leq \varepsilon
\]
for all \( x \in \mathbb{R} \). Furthermore, we get
\[
\| b^k [g(x - k) - ag(x - k - 1)] - b^{k+1}[g(x - k - 1) - ag(x - k - 2)] \| \leq |b|^k \varepsilon
\]  (3.3)
for all \( x \in \mathbb{R} \) and \( k \in \mathbb{Z} \). By (3.3), we obviously have
\[
\| g(x) - ag(x - 1) - b^n [g(x - n) - ag(x - n - 1)] \|
\leq \sum_{k=0}^{n-1} \| b^k [g(x - k) - ag(x - k - 1)]
\]
\[
- b^{k+1}[g(x - k - 1) - ag(x - k - 2)] \| \leq \sum_{k=0}^{n-1} |b|^k \varepsilon
\]  (3.4)
for \( x \in \mathbb{R} \) and \( n \in \mathbb{N} \).

For any \( x \in \mathbb{R} \), (3.3) implies that the sequence \( \{b^n [g(x - n) - ag(x - n - 1)]\} \) is a Cauchy sequence. (Note that \( 0 < |b| < 1 \). Therefore, we can define a function \( G_1 : \mathbb{R} \to X \) by
\[
G_1(x) := \lim_{n \to \infty} b^n [g(x - n) - ag(x - n - 1)],
\]
since \( X \) is complete. In view of the definition of \( G_1 \) and using the relations, \( a + b = \alpha + p \) and \( ab = -r/\alpha \), we obtain
\[
(\alpha + p)G_1(x - 1) + \frac{r}{\alpha}G_1(x - 2) = (a + b)G_1(x - 1) - abG_1(x - 2)
\]
\[
= \underbrace{\frac{a + b}{b}} \lim_{n \to \infty} b^{n+1}[g(x - (n + 1)) - ag(x - (n + 1) - 1)]
\]
\[
- \frac{ab}{b^n} \lim_{n \to \infty} b^{n+2}[g(x - (n + 2)) - ag(x - (n + 2) - 1)]
\]  (3.5)
\[
= \frac{a + b}{b} G_1(x) - \frac{a}{b} G_1(x)
\]
\[
= G_1(x)
\]
for all \( x \in \mathbb{R} \). Since \( \alpha \) is a nonzero root of the cubic equation (2.1), it follows from (3.5) that
\[
G_1(x) - pG_1(x - 1) - qG_1(x - 2) - rG_1(x - 3)
\]
\[
= (\alpha + p)G_1(x - 1) + \frac{r}{\alpha}G_1(x - 2) - pG_1(x - 1) - qG_1(x - 2) - rG_1(x - 3)
\]
\[
= \alpha G_1(x - 1) + \left( -q + \frac{r}{\alpha} \right)G_1(x - 2) - rG_1(x - 3)
\]
\[
= \alpha G_1(x - 1) + (-\alpha^2 - p\alpha)G_1(x - 2) - rG_1(x - 3)
\]
\[
= \alpha \left( (\alpha + p)G_1(x - 2) + \frac{r}{\alpha}G_1(x - 3) \right) - \alpha (\alpha + p)G_1(x - 2) - rG_1(x - 3)
\]
\[
= 0
\]
for all $x \in \mathbb{R}$. Hence, we conclude that $G_1$ is a solution of (1.5).

If $n$ tends to infinity, then (3.4) yields that

$$\|g(x) - ag(x - 1) - G_1(x)\| \leq \frac{\varepsilon}{1 - |b|} \quad (3.6)$$

for every $x \in \mathbb{R}$.

On the other hand, it also follows from (3.1) that

$$\|g(x) - bg(x - 1) - a[g(x - 1) - bg(x - 2)]\| \leq \varepsilon$$

for all $x \in \mathbb{R}$. Analogously to (3.7), replacing $x$ by $x + k$ in the last inequality and then dividing by $|a|^k$ both sides of the resulting inequality, then we have

$$\|a^{-k}[g(x + k) - bg(x + k - 1)] - a^{-k+1}[g(x + k - 1) - bg(x + k - 2)]\| \leq |a|^{-k} \varepsilon \quad (3.7)$$

for all $x \in \mathbb{R}$ and $k \in \mathbb{Z}$. By using (3.7), we further obtain

$$\|a^{-n}[g(x + n) - bg(x + n - 1)] - [g(x) - bg(x - 1)]\| \leq \sum_{k=1}^{n} \|a^{-k}[g(x + k) - bg(x + k - 1)] - a^{-k+1}[g(x + k - 1) - bg(x + k - 2)]\| \leq \sum_{k=1}^{n} |a|^{-k} \varepsilon \quad (3.8)$$

for $x \in \mathbb{R}$ and $n \in \mathbb{N}$.

On account of (3.7), we see that the sequence $\{a^{-n}[g(x + n) - bg(x + n - 1)]\}$ is a Cauchy sequence for any fixed $x \in \mathbb{R}$. (Note that $|a| > 1$). Hence, we can define a function $G_2 : \mathbb{R} \rightarrow X$ by

$$G_2(x) := \lim_{n \rightarrow \infty} a^{-n}[g(x + n) - bg(x + n - 1)].$$

Due to the definition of $G_2$ and the relations, $a + b = \alpha + p$ and $ab = -r/\alpha$, we get

$$(\alpha + p)G_2(x - 1) + \frac{r}{\alpha}G_2(x - 2) = (a + b)G_2(x - 1) - abG_2(x - 2)$$

$$= \frac{a + b}{a} \lim_{n \rightarrow \infty} a^{-(n-1)}[g(x + n - 1) - bg(x + n - 2)] - \frac{ab}{a} \lim_{n \rightarrow \infty} a^{-(n-2)}[g(x + n - 2) - bg(x + n - 3)]$$

$$= \frac{a + b}{a} G_2(x) - \frac{b}{a} G_2(x)$$

$$= G_2(x)$$

for any $x \in \mathbb{R}$. Similarly as in the first part, we can show that $G_2$ is a solution of Eq. (1.5).

If we let $n$ tend to infinity, then it follows from (3.8) that

$$\|G_2(x) - g(x) + bg(x - 1)\| \leq \frac{\varepsilon}{|a| - 1} \quad (3.9)$$

for $x \in \mathbb{R}$. 

It follows from (3.6) and (3.9) that
\[
\left\| g(x - 1) - \frac{1}{a - b} G_2(x) + \frac{1}{a - b} G_1(x) \right\|
\leq \left\| \frac{1}{a - b} G_1(x) - \frac{1}{a - b} g(x) + \frac{a}{a - b} g(x - 1) \right\|
+ \left\| \frac{1}{a - b} g(x) - \frac{b}{a - b} g(x - 1) - \frac{1}{a - b} G_2(x) \right\|
\leq \frac{|a| - |b|}{|a - b| (|a| - 1)(1 - |b|)} \varepsilon
\]
for any \( x \in \mathbb{R} \).

Finally, if we define a function \( G : \mathbb{R} \to X \) by
\[
G(x) := \frac{1}{a - b} G_2(x + 1) - \frac{1}{a - b} G_1(x + 1)
\]
for all \( x \in \mathbb{R} \), then \( G \) is also a solution of Eq. (1.5). Moreover, the validity of (3.2) follows from the last inequality. \( \square \)

The following theorem is the main theorem of this paper.

**Theorem 3.2** Given real constants \( p, q, r \) with \( r \neq 0 \), let \( \alpha_1 \) and \( \alpha_2 \) be distinct nonzero roots of the cubic equation (2.1) and let \( a_i, b_i \) be the roots of the quadratic equation \( x^2 - (\alpha_i + p)x - r/\alpha_i = 0 \) with \( |a_i| > 1 \) and \( 0 < |b_i| < 1 \) for \( i \in \{1, 2\} \). Assume that either \( (\alpha_i + p)^2 + 4r/\alpha_i > 0 \) for all \( i \in \{1, 2\} \) or \( (\alpha_i + p)^2 + 4r/\alpha_i < 0 \) for all \( i \in \{1, 2\} \). Let \( X \) be either a real Banach space if \( (\alpha_i + p)^2 + 4r/\alpha_i > 0 \) or a complex Banach space if \( (\alpha_i + p)^2 + 4r/\alpha_i < 0 \). If a function \( f : \mathbb{R} \to X \) satisfies the inequality (3.1) for all \( x \in \mathbb{R} \) and for some \( \varepsilon \geq 0 \), then there exists a solution \( F : \mathbb{R} \to X \) of Eq. (1.5) such that
\[
\| f(x) - F(x) \| \leq \frac{|a_1| - |b_1|}{|a_1 - b_1|} \frac{|a_2|}{|a_1 - a_2| (|a_1| - 1)(1 - |b_1|)} \varepsilon
+ \frac{|a_2| - |b_2|}{|a_2 - b_2|} \frac{|a_1|}{|a_1 - a_2| (|a_2| - 1)(1 - |b_2|)} \varepsilon
\]
(3.10)
for all \( x \in \mathbb{R} \).

**Proof.** According to Theorem 3.1, there exists a solution \( F_i : \mathbb{R} \to X \) of Eq. (1.5) such that
\[
\| f(x) + \alpha_i f(x - 1) - F_i(x) \| \leq \frac{|a_1| - |b_i|}{|a_i - b_i| (|a_i| - 1)(1 - |b_i|)} \varepsilon
\]

for any \( x \in \mathbb{R} \) and \( i \in \{1, 2\} \). In view of the last inequalities, we have

\[
\left\| f(x) - \frac{\alpha_1}{\alpha_1 - \alpha_2} F_2(x) + \frac{\alpha_2}{\alpha_1 - \alpha_2} F_1(x) \right\| \\
\leq \left\| \frac{\alpha_2}{\alpha_1 - \alpha_2} F_1(x) - \frac{\alpha_2}{\alpha_1 - \alpha_2} f(x) - \frac{\alpha_1 \alpha_2}{\alpha_1 - \alpha_2} f(x - 1) \right\| \\
+ \left\| \frac{\alpha_1}{\alpha_1 - \alpha_2} f(x) + \frac{\alpha_1 \alpha_2}{\alpha_1 - \alpha_2} f(x - 1) - \frac{\alpha_1}{\alpha_1 - \alpha_2} F_2(x) \right\| \\
\leq \frac{|a_1| - |b_1|}{|a_1 - b_1|} \frac{|\alpha_1| - |\alpha_2|}{|\alpha_1 - \alpha_2|} (|a_1| - 1)(1 - |b_1|) \\
+ \frac{|a_2| - |b_2|}{|a_2 - b_2|} \frac{|\alpha_1|}{|\alpha_1 - \alpha_2|} (|a_2| - 1)(1 - |b_2|)
\]

for all \( x \in \mathbb{R} \).

If we define a function \( F : \mathbb{R} \to X \) by

\[
F(x) := \frac{\alpha_1}{\alpha_1 - \alpha_2} F_2(x) - \frac{\alpha_2}{\alpha_1 - \alpha_2} F_1(x)
\]

for each \( x \in \mathbb{R} \), then \( F \) is also a solution of Eq. (1.5), and the inequality (3.10) follows from the last inequality. \( \square \)

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**References**


