ASYMPTOTICS FOR MOMENTS OF CERTAIN COTANGENT SUMS

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Abstract. In this paper we improve a result on the order of magnitude of certain cotangent sums associated to the Estermann and the Riemann zeta functions.

1. Introduction

The authors in joint work [11] and the second author in his thesis [14], investigated the distribution of cotangent sums

$$c_0\left(\frac{r}{b}\right) = -\sum_{m=1}^{b-1} \frac{m}{b} \cot\left(\frac{\pi m r}{b}\right)$$

as r ranges over the set

$$\{r \,:\, (r,b)=1,\ A_0 b \leq r \leq A_1 b\}\,,$$

where A_0, A_1 are fixed with $1/2 < A_0 < A_1 < 1$ and b tends to infinity.
We shall now briefly demonstrate the significance of these sums by exhibiting their relation to other important functions in Mathematics, such as the Estermann and the Riemann zeta functions, as well as their connections to major open problems, such as the Riemann Hypothesis.
These cotangent sums are related to the Estermann zeta function

$$E\left(s, \frac{r}{b}, \alpha\right) := \sum_{n\geq 1} \frac{\sigma_\alpha(n) \exp\left(2\pi i n r/b\right)}{n^s}\,,$$

where $Re\, s > Re\, \alpha + 1$, $b \geq 1$, $(r,b)=1$ and

$$\sigma_\alpha(n) := \sum_{d|n} d^\alpha\,.$$

We have the following result of Ishibashi (see [10]):

Let $b \geq 2$, $1 \leq r \leq b$, $(r,b)=1$, $\alpha \in \mathbb{N} \cup \{0\}$. Then for even α, it holds

$$E\left(0, \frac{r}{b}, \alpha\right) = \left(-\frac{i}{2}\right)^{\alpha+1} \sum_{m=1}^{b-1} \frac{m}{b} \cot^{(\alpha)}\left(\frac{\pi m r}{b}\right) + \frac{1}{4}\delta_{\alpha,0}\,,$$

where $\delta_{\alpha,0}$ is the Kronecker delta function.

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The cotangent sum $c_0(r/b)$ can be associated to the study of the Riemann Hypothesis through its relation with the Vasyunin sum V, which is defined by

$$V\left(\frac{r}{b}\right) := \sum_{m=1}^{b-1} \left\{\frac{mr}{b}\right\} \cot\left(\frac{\pi mr}{b}\right) ,$$

where $\{u\} = u - \lfloor u \rfloor$, $u \in \mathbb{R}$.
It can be shown that

$$V\left(\frac{r}{b}\right) = -c_0\left(\frac{\bar{r}}{b}\right),$$

where $\bar{r}$ is such that $\bar{r}r \equiv 1 \pmod{b}$.
We have (see [5]):
(1)

$$\frac{1}{2\pi(rb)^{1/2}} \int_{-\infty}^{+\infty} \left|\zeta\left(\frac{1}{2}+it\right)\right|^2 \left(\frac{r}{b}\right)^{it} \frac{dt}{\frac{1}{4}+t^2} = \frac{\log 2\pi - \gamma}{2}\left(\frac{1}{r}+\frac{1}{b}\right) + \frac{b-r}{2rb}\log\frac{r}{b} - \frac{\pi}{2rb}\left(V\left(\frac{r}{b}\right)+V\left(\frac{b}{r}\right)\right).$$

The above formula is related to the Nyman-Beurling-Baéz-Duarte-Vasyunin approach to the Riemann Hypothesis (see [2], [5]).
According to this approach, the Riemann Hypothesis is true if and only if

$$\lim_{N\to+\infty} d_N = 0,$$

where

$$d_N^2 := \inf_{D_N} \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left|1-\zeta\left(\frac{1}{2}+it\right) D_N\left(\frac{1}{2}+it\right)\right|^2 \frac{dt}{\frac{1}{4}+t^2} \tag{2}$$

and the infimum is taken over all Dirichlet polynomials

$$D_N(s) := \sum_{n=1}^{N} \frac{a_n}{n^s}.$$

The authors of the present paper in joint work (cf. [12]), considered the moments defined by

$$H_k = \lim_{b\to+\infty} \phi(b)^{-1} b^{-2k}(A_1-A_0)^{-1} \sum_{\substack{A_0 b\le r\le A_1 b\\ (r,b)=1}} c_0\left(\frac{r}{b}\right)^{2k},\ k\in\mathbb{N},$$

where $\phi(\cdot)$ denoted the Euler phi-function.
They could show that

$$H_k = \int_0^1 \left(\frac{g(x)}{\pi}\right)^{2k} dx ,$$

where

$$g(x) = \sum_{l\ge1} \frac{1-2\{lx\}}{l} ,$$

a function that has been investigated by de la Bretèche and Tenenbaum [7, 8], as well as Balazard and Martin [3, 4]. Bettin [6] could replace the interval $(1/2,1)$ for A_0, A_1 by the interval $(0,1)$.

In [12], Theorem 1.1 the authors could determine the order of magnitude of H_k. There are constants c_1, $c_2 > 0$, such that

$$c_1\Gamma(2k+1) \le \int_0^1 g(x)^{2k} dx \le c_2\Gamma(2k+1)\,, \tag{1.1}$$

for all $k \in \mathbb{N}$, where $\Gamma(\cdot)$ stands for the Gamma function.
In this paper we extend the result of (1.1) to an asymptotic formula valid for arbitrary natural exponents.

Theorem 1.1. *Let*

$$A = \int_0^\infty \frac{\{t\}^2}{t^2} dt$$

and $K \in \mathbb{N}$. There is an absolute constant $C > 0$, such that

$$\int_0^1 |g(x)|^K dx = 2e^{-A}\Gamma(K+1)(1 + O(\exp(-CK))),$$

for $K \to \infty$.

2. Overview and preliminary results

Like in the proof of (1.1), a crucial role is played by the relation of $g(x)$ to Wilton's function, established by Balazard and Martin [4] and results about operators related to continued fraction expansions due to Marmi, Moussa and Yoccoz [13].
We recall some fundamental definitions and results.

Definition 2.1. *Let $X = (0,1) \setminus \mathbb{Q}$. Let $\alpha(x) = \{1/x\}$ for $x \in X$. The iterates α_k of α are defined by $\alpha_0(x) = x$ and*

$$\alpha_k(x) = \alpha(\alpha_{k-1}(x)),\ \textit{for } k > 1.$$

Lemma 2.2. *Let $x \in X$ and let*

$$x = [a_0(x); a_1(x), \ldots, a_k(x), \ldots]$$

be the continued fraction expansion of x. We define the partial quotient of $p_k(x)$, $q_k(x)$:

$$\frac{p_k(x)}{q_k(x)} = [a_0(x); a_1(x), \ldots, a_k(x)],\ \textit{where},\ (p_k(x), q_k(x)) = 1\,.$$

Then we have

$$a_k(x) = \left\lfloor \frac{1}{\alpha_{k-1}(x)} \right\rfloor\,,$$

$$p_{k+1} = a_{k+1}p_k + p_{k-1}$$

and

$$q_{k+1} = a_{k+1}q_k + q_{k-1}\,.$$

Proof. This is Lemma 2.2 of [12]. □

Definition 2.3. *Let $x \in X$. Let also*

$$\beta_k(x) = \alpha_0(x)\alpha_1(x)\cdots\alpha_k(x),\ \beta_{-1}(x) = +1$$

$$\gamma_k(x) = \beta_{k-1}(x)\log\frac{1}{\alpha_k(x)},\ \textit{where } k \ge 0,$$

so that $\gamma_0(x) = \log(1/x)$.
The number x *is called a* ***Wilton number*** *if the series*

$$\sum_{k\geq 0}(-1)^k\gamma_k(x)$$

converges.
Wilton's function $\mathcal{W}(x)$ *is defined by*

$$\mathcal{W}(x) = \sum_{k\geq 0}(-1)^k\gamma_k(x)$$

for each Wilton number $x \in (0,1)$.

Lemma 2.4. *A number* $x \in X$ *is a Wilton number if and only if* $\alpha(x)$ *is a Wilton number. In this case we have:*

$$\mathcal{W}(x) = \log\frac{1}{x} - x\mathcal{W}(\alpha(x)).$$

Proof. This is Lemma 2.4 of [12]. □

Definition 2.5. *Let* $p > 1$ *and* $T : L^p \to L^p$ *be defined by*

$$Tf(x) = xf(\alpha(x)).$$

The measure m *is defined by*

$$m(\mathcal{E}) = \frac{1}{\log 2}\int_{\mathcal{E}}\frac{dx}{1+x},$$

where f *is any measurable subset of* $(0,1)$.

Lemma 2.6. *Let* $p > 1$, $n \in \mathbb{N}$.
(i) The measure m *is invariant with respect to the map* α, *i.e.*

$$m(\alpha(\mathcal{E})) = m(\mathcal{E})\ ,$$

for all measurable subsets of $\mathcal{E} \subset (0,1)$.
(ii) For $f \in L^p$ *we have*

$$\int_0^1 |T^n f(x)|^p dm(x) \leq g^{(n-1)p}\int_0^1 |f(x)|^p dm(x),$$

where

$$g := \frac{\sqrt{5}-1}{2} < 1.$$

Proof. This is Lemma 2.8 of [12]. □

Lemma 2.7. *There is a bounded function* $H : (0,1) \to \mathbb{R}$, *which is continuous in every irrational number, such that*

$$g(x) = \mathcal{W}(x) + H(x)\ .$$

Proof. See Lemma 2.5 of [12]. □

Lemma 2.5 of [12] is based on [4]. In the proof of (1.1) we only use the boundedness of H.
The key to the improvement of (1.1) is the use of more subtle properties of H. We recall the following definitions and results from [4].

Definition 2.8. *For $\lambda \geq 0$, we set*

$$A(\lambda) := \int_0^\infty \{t\}\{\lambda t\}\frac{dt}{t^2}\,,$$

$$F(x) := \frac{x+1}{2}A(1) - A(x) - \frac{x}{2}\log x\,,$$

$$G(x) := \sum_{j\geq 0}(-1)^j \beta_{j-1}(x)F(\alpha_j(x))\,,$$

$$B_1(t) := t - \lfloor t \rfloor - 1/2,\ \textit{the first Bernoulli function}\,,$$

$$B_2(t) := \{t\}^2 - \{t\} + 1/6,\ (t \in \mathbb{R})\ \textit{the second Bernoulli function}\,.$$

For $\lambda \in \mathbb{R}$, let

$$\phi_2(\lambda) := \sum_{n\geq 1}\frac{B_2(n\lambda)}{n^2}\,.$$

Lemma 2.9. *It holds*

$$A(\lambda) = \frac{\lambda}{2}\log\frac{1}{\lambda} + \frac{1+A(1)}{2}\,\lambda + O(\lambda^2), \quad \textit{as } \lambda \to 0\,.$$

Proof. By [4], Proposition 31, formula (74), we have:

$$A(\lambda) = \frac{\lambda}{2}\log\frac{1}{\lambda} + \frac{1+A(1)}{2}\,\lambda + \frac{\lambda^2}{2}\phi_2\left(\frac{1}{\lambda}\right) - \int_{1/\lambda}^\infty \phi_2(t)\frac{dt}{t^3}\,.$$

From Definition 2.8, it follows that $\phi_2(t)$ is bounded. Therefore

$$\frac{\lambda^2}{2}\phi_2\left(\frac{1}{\lambda}\right) = O(\lambda^2)$$

and

$$\int_{1/\lambda}^\infty \phi_2(t)\frac{dt}{t^3} = O(\lambda^2).$$

□

Lemma 2.10. *We have*

$$H(x) = 2\sum_{j\geq 0}(-1)^{j-1}\beta_{j-1}(x)F(\alpha_j(x)).$$

Proof. In [4] the function Φ_1 is defined by

$$\Phi_1(t) := \sum_{n\geq 1}\frac{B_1(nt)}{n} = \sum_{n\geq 1}\frac{\{nt\}-1/2}{n}\,. \tag{2.1}$$

Thus we have

$$g(x) = -2\Phi_1(x)\,. \tag{2.2}$$

By Proposition (2) of [4] we obtain

$$\Phi_1(x) = -\frac{1}{2}\mathcal{W}(x) + G(x) \tag{2.3}$$

almost everywhere.

The proof of Lemma 2.10 follows now from Lemma 2.7, (2.1), (2.2) and (2.3) by the choice

$$H = -2G\,. \tag{2.4}$$

□

3. Proof of Theorem 1.1

Definition 3.1. *Let $d, h \in \mathbb{N}_0$, $h \geq 1$, $u, v \in (0, \infty)$. Then we define*

$$\mathcal{J}(d,h,u,v) := \{x \in X \ : \ T^d l(x) \geq u \text{ and } T^{d+h} l(x) \geq v\}\,.$$

Lemma 3.2. *We have*

$$m(\mathcal{J}(d,h,u,v)) \leq 2\exp\left(-2^{\frac{h-2}{2}} v \exp\left(2^{\frac{d-2}{2}} u\right)\right)$$

Proof. This is Lemma 2.13 of [12]. □

Definition 3.3. *For $n \in \mathbb{N}$, $x \in X$, we define*

$$\mathcal{L}(x,n) := \sum_{v=0}^{n} (-1)^v (T^v l)(x)\,,$$

where $l(x) = \log(1/x)$.

Definition 3.4. *(Definition 2.14 of [12])*
We set $j_0 := L - \left\lfloor \frac{L}{100} \right\rfloor$, $C_2 := 1/400$. For $j \in \mathbb{Z}$, $j \leq j_0$, we define the intervals

$$I(L,j) := (\exp(-L+j-1),\, \exp(-L+j))\,.$$

For $v \in \mathbb{N}_0$, we set

$$a(L,v) := \exp(-C_2 L + v).$$

$$\mathcal{T}(L,j,0) := \{x \in I(L,j) \cap X \ : \ |\mathcal{L}(x,n) - l(x)| \leq \exp(-C_2 L)\}\,,$$

and for $v \in \mathbb{N}$, we set

$$\mathcal{T}(L,j,v) := \{x \in I(L,j) \cap X \ : \ a(L,v-1) \leq |\mathcal{L}(x,n) - l(x)| \leq a(L,v)\}\,.$$

For $v, h \in \mathbb{Z}$, $v \geq 1$, $h \geq 0$, we set

$$U(L,j,v,h) := \{x \in \mathcal{T}(L,j,v) \ : \ T^h l(x) \geq 2^{-h} a(L,v-1)\}\,.$$

Lemma 3.5. *There are constants $C_3, C_4 > 0$, such that for $v \geq 1$, we have*

$$m(\mathcal{T}(L,j,v)) \leq C_3 \exp\left(-C_4 \exp\left(-C_2 L + v - 1 + \frac{1}{2}(L-j)\right)\right)\,.$$

Proof. This is lemma 2.15 of [12]. □

Definition 3.6. *(Definition 2.16 of [12])*
We set

$$x_0 := \exp\left(-\left\lfloor \frac{L}{100} \right\rfloor\right)\,.$$

Lemma 3.7. *Let $L \in \mathbb{N}$, then*
(i)

$$\int_0^1 l(x)^L dx = \Gamma(L+1)$$

(ii) There is a constant $C_5 > 0$, such that

$$\int_{x_0}^1 l(x)^L dx = O(\Gamma(L+1)\exp(-C_5 L))\,.$$

Proof. This is parts (i) and (ii) of Lemma 2.17 of [12]. □

Lemma 3.8. *Let $1 < p \leq 2$, such that $pL \in \mathbb{N}$. There is $n_0 \in \mathbb{N}$ and a constant $C_6 > 0$, such that for $n \geq n_0$, we have:*

$$\int_0^{x_0} |\mathcal{L}(x,n)^L - l(x)^L|^p dm(x) \leq \Gamma(pL+1)\exp(-C_6 L) .$$

Proof. We write

$$\mathcal{L}(x,n) := l(x)(1 + R(x,n)) . \tag{3.1}$$

Let $j \leq j_0$. Then by Definition 3.4, for $x \in \mathcal{T}(L,j,v)$ we have $l(x) \geq L - j$ and therefore we get

$$l(x) \geq \frac{L}{200} . \tag{3.2}$$

By Definition 3.4 we also have

$$|\mathcal{L}(x,n) - l(x)| \leq \exp(-C_2 L + v) . \tag{3.3}$$

From (3.2) and (3.3) we have:

$$|R(x,n)| \leq \frac{200}{L}\exp(-C_2 L + v) . \tag{3.4}$$

We distinguish two cases:
Case 1: Let $v = 0$.
From (3.4) we have

$$|R(x,n)| \leq \exp\left(-\frac{C_2}{2}L\right) . \tag{3.5}$$

$$\int_{\mathcal{T}(L,j,0)} |\mathcal{L}(x,n)^L - l(x)^L|^p dx \leq \int_{\mathcal{T}(L,j,0)} l(x)^{pL}|(1+R(x,n))^L - 1|^p dx \tag{3.6}$$

From (3.5) and (3.6) we have:

$$\int_{\mathcal{T}(L,j,0)} |\mathcal{L}(x,n)^L - l(x)^L|^p dx \leq \exp\left(-\frac{C_2}{3}L\right)\int_{\mathcal{T}(L,j,0)} l(x)^{pL} dx . \tag{3.7}$$

Case 2: Let $v \geq 1$.
Because of the fact that

$$L - j \geq \frac{L}{100} ,$$

we have for an appropriate constant $C_7 > 0$ that

$$\max_{x\in I(L,j)} l(x)^L \leq C_7 \min_{x \in I(L,j)} l(x)^L$$

and therefore from (3.4), it follows that

(3.8)

$$\int_{\mathcal{T}(L,j,v)} |\mathcal{L}(x,n)^L - l(x)^L|^p dx \leq \exp(-C_2 L + v) m(\mathcal{T}(L,j,v)) \max_{x\in I(L,j)} l(x)^{pL}$$

$$\leq C_3 C_7 \exp\left(-C_4 \exp\left(-C_2 L + v - 1 + \frac{1}{2}(L-j)\right)\right)\exp(-C_2 L + v) \min_{x\in I(L,j)} l(x)^{pL} .$$

From (3.7) and (3.8), we obtain for $j \leq j_0$, the following

$$\int_{I(L,j)\cap X} |\mathcal{L}(x,n)^L - l(x)^L|^p dx \leq \exp\left(-\frac{C_2}{3}L\right)\int_{I(L,j)} l(x)^{pL} dx \tag{3.9}$$

The result of Lemma 3.8 now follows from Lemma 3.7 by summing (3.9) for $j \le j_0$. □

Lemma 3.9. *Let $1 < p \le 2$ and $pL \in \mathbb{N}$. There is a constant $C_8 > 0$, such that*

$$\int_{x_0}^{1/2} |\mathcal{L}(x,n)|^{pL} dx \le \Gamma(pL+1)\exp(-C_8 L) .$$

Proof. Lemma 3.9 follows if we apply Lemma 2.22 from [12] with pL instead of L. □

Lemma 3.10. *Let $0 < \alpha < 1$. Then, there is a constant $C = C(\alpha) > 0$, such that*

$$\int_0^{1/2} x^\alpha l(x)^L dx \le \Gamma(L+1)\exp(-CL) ,$$

for all $L \in \mathbb{N}$.

Proof. We have

$$\int_0^{1/2} x^\alpha l(x)^L dx \le \sum_{0\le j\le j_0} \int_{I(L,j)} x^\alpha l(x)^L dx + \int_{x_0}^{1/2} x^\alpha l(x)^L dx .$$

For $x \in I(L,j) = (\exp(-L+j-1),\ \exp(-L+j))$ we have $l(x) \le L-(j-1)$ and therefore

$$l(x)^L = O(L^L e^{-j}).$$

Therefore, by Stirling's formula

$$\begin{aligned}\int_{I(L,j)} x^\alpha l(x)^L dx &= O(L^L \exp((\alpha+1)(-L+j)-j) \\ &= O(\Gamma(L+1)\exp(-\alpha L+(\alpha-1)j+\epsilon L),\end{aligned}$$

for all $\epsilon > 0$, which proves Lemma 3.10. □

Lemma 3.11. *For $m \in \mathbb{N}_0$, $x \in X$, we have*

$$\alpha_m(x)\alpha_{m+1}(x) \le \frac{1}{2} .$$

Proof. This is Lemma 2.11 of [12]. □

Definition 3.12. *For $l_1, l_2 \in \mathbb{N}_0$, $0 \le l_1 + l_2 \le K$, we set*

$$\int_{(l_1,l_2)} := \int_0^{1/2} \mathcal{L}(x,n)^{K-l_1-l_2} H(x)^{l_1}((-1)^{n+1}T^{n+1}\mathcal{W}(x))^{l_2} dx .$$

Lemma 3.13. *There is a constant $C_9 > 0$, such that*

$$\int_0^{1/2} |g(x)^K - |g(x)|^K| dx \le \Gamma(K+1)\exp(-C_9 K) .$$

Proof. Let

$$x \in I(K,j) = (\exp(-K+j-1, \exp(-K+j)) .$$

Let

$$\mathcal{Y}(K,j) = \{x \in I(K,j)\ :\ g(x) \le 0\} .$$

For $x \in \mathcal{Y}(K,j)$ we must have

$$x \in \mathcal{T}(K,j,v) \ \text{ for } v \ge C_2 K \ \text{ or } |T^n \mathcal{W}(x)| \ge K - j - H , \tag{3.10}$$

where

$$H = \sup_{x\in[0,1]} |H(x)| \,.$$

For $w \in \mathbb{N}$, let

(3.11) $\mathcal{V}(K,j,w,n) = \{x \in I(L,j) : L{-}j{-}H{+}w \le |T^n\mathcal{W}(x)| \le L{-}j{-}H{+}w{+}1\}.$

Let

$$\mathcal{Z}(K,j,w,n) = \mathcal{T}(K,j,v) \cap \mathcal{V}(K,j,w,n) \,. \tag{3.12}$$

By Lemma 2.6 (ii) we have:

$$m(\mathcal{V}(K,j,w,n))(K{-}j{-}H{+}w)^2 \le \int_{\mathcal{V}(L,j,w)} |T^n\mathcal{W}(x)|^2 dm(x) \le g^{2(n-1)} \int_0^1 |\mathcal{W}(x)|^2 dm(x).$$

Thus

$$m(\mathcal{V}(K,j,w,n)) \le g^{2(n-1)} \int_0^1 |\mathcal{W}(x)|^2 dm(x)\, (L-j-H+w)^{-2} \,. \tag{3.13}$$

We have

$$|g(x)^K - |g(x)|^K| \le 2|g(x)|^K \tag{3.14}$$

and for $x \in \mathcal{Z}(K,j,w,n)$

$$|g(x)| \le b(x,K,j,n) + |\mathcal{L}(x,n) - l(x)|, \tag{3.15}$$

where $b(x,K,j,n) := l(x) + L - j + w + 1$. Thus, from (3.14) we get

(3.16)

$$\int_{\mathcal{Z}(K,j,w,n)} |g(x)^K - |g(x)|^K| dx \le 2^K \left(\sup_{x\in I(K,j)} |b(x,K,j,n|^K + \int_{I(K,j)} |\mathcal{L}(x,n) - l(x)|^K dx \right) \times \left(m(\mathcal{T}(K,j,v)) + m(\mathcal{V}(K,j,w,n)) \right) \,.$$

From Lemma 3.5, Lemma 3.8, (3.15), (3.16) we get by summation over j, v and w:

$$\int_0^{x_0} |g(x)^K - |g(x)|^K| dx \le \Gamma(K+1)\exp(-C_{10}K), \tag{3.17}$$

where $x_0 := x_0(K) = \exp(-\left\lfloor \frac{K}{100} \right\rfloor)$. From Lemma 3.5, we obtain:

$$\int_{x_0}^{1/2} |g(x)^K - |g(x)|^K| dx \le \Gamma(K+1)\exp(-C_{11}K) \,. \tag{3.18}$$

Lemma 3.13 now follows from (3.17) and (3.18). □

Lemma 3.14. *We have*

$$\int_0^{1/2} g(x)^K dx = \sum_{\substack{(l_1,l_2)\in\mathbb{N}_0^2 \\ 0\le l_1+l_2\le K}} \frac{K!}{(K-l_1-l_2)!\, l_1!\, l_2!} \int_{(l_1,l_2)} \,.$$

Proof. From formula (3) of [12] we have:

$$\mathcal{W}(x) = \mathcal{L}(x,n) + (-1)^{n+1} T^{n+1}\mathcal{W}(x) \,.$$

By Lemma 2.7, we obtain

$$g(x) = \mathcal{L}(x,n) + H(x) + (-1)^{n+1} T^{n+1}\mathcal{W}(x) \,.$$

Lemma 3.14 now follows by the Multinomial Theorem. □

Definition 3.15. *For (l_1, l_2) as in Definition 3.12 we set*

$$\int_{(l_1,l_2)}^{(1)} := \int_0^{1/2} l(x)^{K-l_1-l_2} H(x)^{l_1} [(-1)^{n+1} T^{n+1} l(x)]^{l_2} dx$$

$$\int_{(l_1,l_2)}^{(2)} := \int_0^{1/2} (\mathcal{L}(x,n)^{K-l_1-l_2} - l(x)^{K-l_1-l_2}) H(x)^{l_1} [(-1)^{n+1} T^{n+1} \mathcal{W}(x)]^{l_2} dx \, .$$

Lemma 3.16.

$$\int_{(l_1,l_2)} = \int_{(l_1,l_2)}^{(1)} + \int_{(l_1,l_2)}^{(2)} \, .$$

Proof. Obvious. □

We now show, that the integrals $\int_{(l_1,l_2)}^{(2)}$ for all l_1, l_2 and $\int_{(l_1,l_2)}^{(1)}$, if $l_2 > 0$ are negligible.

Lemma 3.17. *There is an $n_0 = n_0(K) \in \mathbb{N}$, such that for $n \geq n_0$ we have for $i = 1, 2$ and all $l_1 \leq K$ and $l_2 > 0$ the following*

$$\int_{(l_1,l_2)}^{(i)} \leq (K(2K)!)^{-1} \, .$$

Proof. We choose $1 < p \leq 2$. We set $L = K - l_1 - l_2$ and apply Lemma 3.8 with $p = 2$ to obtain from the inequality of Cauchy-Schwarz:

$$\int_{(l_1,l_2)}^{(i)} \leq \left(\int_0^{1/2} I(x)^2 dx \right)^{1/2} \left(\int_0^{1/2} |T^{n+1} \mathcal{W}(x)|^{2l_2} dx \right)^{1/2} \sup_{x \in [0,1/2)} |H(x)^{l_2}| \, ,$$

where

$$I(x) := l(x)^L, \text{ for } i = 1$$

and

$$I(x) := \mathcal{L}(x,n)^L - l(x)^L, \quad \text{for } i = 2.$$

By Lemma 2.6 we obtain the result if we choose n_0 sufficiently large. □

Lemma 3.18. *Assume L_0 is sufficiently large and that $L := K - l_1 \geq L_0$. There are constants $C_9, C_{10} > 0$, such that*

$$\left| \int_{(l_1,0)}^{(2)} \right| \leq C_0^{l_1} K^{l_1} \Gamma(K + 1 - l_1) \exp(-C_{13} K) \, .$$

Proof. Let $|H(x)| \leq C_{11}$ with $C_{11} > 0$. We choose p, $1 < p \leq 2$, such that $pL \in \mathbb{N}$. We define $\epsilon > 0$ by $(1-\epsilon)^{-1} = p$. Then by Lemma 3.8 and Hölder's inequality we have

$$\begin{aligned} \int_{(l_1,0)}^{(2)} &\leq \left(\int_0^{1/2} |\mathcal{L}(x,n)^L - l(x)^L|^p dx \right)^{1/p} \left(\int_0^{1/2} |H(x)|^{l_1/\epsilon} dx \right)^{\epsilon} \\ &\leq \Gamma(pL+1)^{1/p} \exp\left(-\frac{C_6}{p} L \right) C_{11}^{l_1} \, . \end{aligned}$$

By Stirling's formula

$$\int_{(l_1,l_2)}^{(2)} \leq (pL)^L \exp\left(-\frac{L - 3\epsilon}{p} \right) \exp\left(-\frac{C_6}{p} L \right)$$

for sufficiently large L.
Since $\epsilon \to 0$ for $L \to \infty$, the result of Lemma 3.18 follows. $\square$

Lemma 3.19. *There is a constant $C_{15} > 0$, such that*

$$\int_0^{1/2} g(x)^K dx = \sum_{0 \le l_1 \le K} \binom{K}{l_1} \int_{(l_1,0)}^{(1)} + O(\Gamma(K+1)\exp(-C_{15}K)) \, .$$

Proof. This follows from Lemms 3.16 - 3.18. $\square$

Definition 3.20. *Let $0 \le m \le l_1$. Then we set*

$$\int^{(l_1,m)} := \int_0^{1/2} l(x)^{2k-l_1}(-2F(x))^{l_1-m}\left(\sum_{j>0}(-1)^{j-1}\beta_{j-1}F(\alpha_j(x))\right)^m dx \, .$$

Lemma 3.21.

$$\int_{(l_1,0)}^{(1)} = \sum_{m=0}^{l_1}\binom{l_1}{m}\int^{(l_1,m)} \, .$$

Proof. This follows from Lemma 3.11, Definition 3.15, 3.20 and the Binomial Theorem. $\square$

Lemma 3.22. *There is a constant $C_{13} > 0$, such that*

$$\int_0^{1/2} g(x)^K dx = \sum_{0 \le l_1 \le K}\binom{K}{l_1}\int^{(l_1,0)} + O\left(\Gamma(K+1)\exp(-C_{13}K)\right) \, .$$

Proof. Let $m > 0$. We have

$$\beta_{j-1} = x\alpha_1(x)\cdots\alpha_{j-1}(x) \, .$$

By Lemma 3.11 we have for an absolute constant C_{14} the following

$$\left|\sum_{j>0}(-1)^{j-1}\beta_{j-1}F(\alpha_j(x))\right| < C_{14}x, \text{ if } x \in (0,1) \, .$$

We also have

$$|-2F(x)| < C_{15} \, .$$

By Lemma 3.10, we therefore have

$$\begin{aligned}\left|\int^{(l_1,m)}\right| &\le C_{15}^{l_1-m}\left|\int_0^{1/2} l(x)^{K-l_1}\left(\sum_{j>0}(-1)^{j-1}\beta_{j-1}F(\alpha_j(x))\right)^m\right| dx \\ &\le \Gamma(K-l_1+1)(3C_{14}C_{15})^{l_1} \, .\end{aligned}$$

Lemma 3.22 follows by summation over l_1. $\square$

Definition 3.23. *For $0 \le l_1 \le K$ we set*

$$Int(l_1) := \int_0^{1/2} l(x)^{K-l_1}(-2F(x))^{l_1} dx \, .$$

For $0 \le m \le l_1$ we set

$$Int(l_1,m) := \int_0^{1/2} l(x)^{K-l_1}(-A(1))^{l_1-m}R(x)^m dx \, ,$$

where
$$R(x) := -xA(1) + A(x) + \frac{x}{2}\log x\,.$$

Lemma 3.24. *We have*
$$Int(l_1) = \sum_{m=0}^{l_1} \binom{l_1}{m} Int(l_1, m)\,.$$

Proof. This follows by Definition 3.23 and the Binomial Theorem. □

Lemma 3.25. *There is a constant $C_{16} > 0$, such that*
$$\int_0^{1/2} g(x)^K dx = \sum_{0\le l_1\le K} \binom{K}{l_1} \int_0^{1/2} l(x)^{K-l_1}(-A(1))^{l_1} dx + O(\Gamma(K+1)\exp(-C_{18}K)).$$

Proof. This follows in a similar manner as the result of Lemma 3.22 by application of Lemma 3.10 and summation over l_1. □

4. Conclusion of the proof of Theorem 1.1

We have
$$\binom{K}{l_1} \int_0^1 l(x)^{K-l_1} dx = \binom{K}{l_1}\Gamma(K-l_1+1) = \frac{1}{l_1!}\Gamma(K+1)\,.$$
From Lemmas 3.7 and 3.25, we therefore get
$$\int_0^{1/2} g(x)^K dx = \left(\sum_{l_1=0}^{\infty} \frac{1}{l_1!}(-A(1))^{l_1}\right)\Gamma(K+1) + O(\Gamma(K+1)\exp(-C_{18}K))\,. \tag{4.1}$$
From Lemma 3.13 and (4.1) we obtain
$$\int_0^{1/2} |g(x)|^K dx = \left(\sum_{l_1=0}^{\infty} \frac{1}{l_1!}(-A(1))^{l_1}\right)\Gamma(K+1) + O(\Gamma(K+1)\exp(-C_{19}K)). \tag{4.2}$$
Since
$$\int_0^{1/2} |g(x)|^K dx = \int_{1/2}^1 |g(x)|^K dx\,,$$
this concludes the proof of Theorem 1.1.

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