John Forbes Nash, Jr. • Michael Th. Rassias
Editors

Open Problems in Mathematics

Springer

Preface

John Forbes Nash, Jr. and Michael Th. Rassias

Learn from yesterday, live for today, hope for tomorrow.
The important thing is not to stop questioning.
– Albert Einstein (1879–1955)

It has become clear to the modern working mathematician that no single researcher, regardless of his knowledge, experience, and talent, is capable anymore of overviewing the major open problems and trends of mathematics in its entirety. The breadth and diversity of mathematics during the last century has witnessed an unprecedented expansion.

In 1900, when David Hilbert began his celebrated lecture delivered before the International Congress of Mathematicians in Paris, he stoically said:

> Who of us would not be glad to lift the veil behind which the future lies hidden; to cast a glance at the next advances of our science and at the secrets of its development during future centuries? What particular goals will there be toward which the leading mathematical spirits of coming generations will strive? What new methods and new facts in the wide and rich field of mathematical thought will the new centuries disclose?

Perhaps Hilbert was among the last great mathematicians who could talk about mathematics as a whole, presenting problems which covered most of its range at the time. One can claim this, not because there will be no other mathematicians of Hilbert's caliber, but because life is probably too short for one to have the opportunity to expose himself to the allness of the realm of modern mathematics. Melancholic as this thought may sound, it simultaneously creates the necessity and aspiration for intense collaboration between researchers of different disciplines. Thus, overviewing open problems in mathematics has nowadays become a task which can only be accomplished by collective efforts.

The scope of this volume is to publish invited survey papers presenting the status of some essential open problems in pure and applied mathematics, including old and new results as well as methods and techniques used toward their solution. One expository paper is devoted to each problem or constellation of related problems. The present anthology of open problems, notwithstanding the fact that it ranges over a variety of mathematical areas, does not claim by any means to be complete, as such a goal would be impossible to achieve. It is, rather, a collection of beautiful mathematical questions which were chosen for a variety of reasons. Some were chosen for their undoubtable importance and applicability, others because they constitute intriguing curiosities which remain unexplained mysteries on the basis of current knowledge and techniques, and some for more emotional reasons. Additionally, the attribute of a problem having a somewhat *vintage flavor* was also influential in our decision process.

The book chapters have been contributed by leading experts in the corresponding fields. We would like to express our deepest thanks to all of them for participating in this effort.

Princeton, NJ, USA
April, 2015

John F. Nash, Jr.

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A Farewell to "A Beautiful Mind and a Beautiful Person"

Michael Th. Rassias

Having found it very hard to resign myself to John F. Nash's sudden and so tragic passing, I postponed writing my commemorative addendum to our jointly composed preface until this compilation of papers on open problems was almost fully ready for publication. Now that I have finally built up my courage for coming to terms with John Nash's demise, my name, which joyfully adjoins his at the end of the above preface, now also stands sadly alone below the following bit of reminiscence from my privileged year as his collaborator and frequent companion.

It all started in September 2014, in one of the afternoon coffee/tea meetings that take place on a daily basis in the common room of Fine Hall, the building housing the Mathematics Department of Princeton University. John Nash silently entered the room, poured himself a cup of decaf coffee and then sat alone in a chair close by. That was when I first approached him and had a really pleasant chat about problems in the interplay of game theory and number theory. From that day onwards, our discussions became ever more frequent, and we eventually decided to prepare this volume *Open Problems in Mathematics*. The day we made this decision, he turned to me and said with his gentle voice, "I don't want to be *just a name* on the cover though. I want to be really involved." After that, we met almost daily and discussed for several hours at a time, examining a vast number of open problems in mathematics ranging over several areas. During these discussions, it became even clearer to me that his way of thinking was very different from that of almost all other mathematicians I have ever met. He was thinking in an unconventional, most creative way. His quick and distinctive mind was still shining bright in his later years.

This volume was practically almost ready before John and Alicia Nash left in May for Oslo, where he was awarded the 2015 Abel Prize from the Norwegian Academy of Science and Letters. We had even prepared the preface of this volume, which he was so much looking forward to see published. Our decision to include handwritten signatures, as well, was along the lines of the somewhat vintage flavor and style that he liked.

John Nash was planning to write a brief article on an open problem in game theory, which was the only problem we had not discussed yet. He was planning to prepare it and discuss about it after his trip to Oslo. Thus, he never got the opportunity to write it. On this note, and notwithstanding my ‘last-minute’ invitation, Professor Eric Maskin generously accepted to contribute a paper presenting an important open problem in cooperative game theory.

With this opportunity, I would also like to say just a few words about the man behind the mathematician. In the famous movie *A Beautiful Mind*, which portrayed his life, he was presented as a really combative person. It is true that in his early years he might have been, having also to battle with the demons of his illness. Being almost 60 years younger than him, I had the chance to get acquainted with his personality in his senior years. All the people who were around him, including myself, can avow that he was a truly wonderful person. Very kind and disarmingly simple, as well as modest. This is the reason why, among friends at Princeton, I used to humorously say that the movie should have been called *A Beautiful Mind and a Beautiful Person*. What was certainly true though was the dear love between John and Alicia Nash, who together faced and overcame the tremendous challenges of John Nash’s life. It is somehow a romantic tragedy that fate bound them to even leave this life together.

In history, one can say that among the mathematicians who have reached greatness, there are some—a selected few—who have gone beyond greatness to become legends. John Nash was one such legend.

The contributors of papers and myself cordially dedicate this volume to the memory and rich mathematical legacy of John F. Nash, Jr.

Princeton, NJ, USA Michael Th. Rassias

Introduction
John Nash: Theorems and Ideas

Misha Gromov

Nash was not building big theories, he did not attempt to dislodge old concepts and to promote new ones, he didn't try to be paradoxical.

Nash was solving classical mathematical problems, difficult problems, something that nobody else was able to do, not even to imagine how to do it.

His landmark theorem of 1956—one of the main achievements of mathematics of the twentieth century–reads:

> All Riemannian manifolds X can be realised as smooth submanifolds in Euclidean spaces $\mathbb{R}^q$, such that the smoothness class of the submanifold realising an X in $\mathbb{R}^q$ equals that of the Riemannian metric g on X and where the dimension q of the ambient Euclidean space can be universally bounded in terms of the dimension of X.[1]

And as far as C^1-*smooth* isometric embeddings $f : X \to \mathbb{R}^q$ are concerned, there is no constraint on the dimension of the Euclidean space except for what is dictated by the topology of X:

> Every C^1-smooth n-dimensional submanifold X_0 in $\mathbb{R}^q$ for $q \geq n+1$ can be deformed (by a C^1-isotopy) to a new C^1-position such that the induced Riemannian metric on X_0 becomes equal to a given g.[2]

At first sight, these are natural classically looking theorems. But what Nash has discovered in the course of his constructions of isomeric embeddings is far from "classical"—it is something that brings about a dramatic alteration of our understanding of the basic logic of analysis and differential geometry. Judging from

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[1]This was proven in the 1956 paper for C^r-smooth metrics, $r = 3, 4, \ldots, \infty$; the existence of *real analytic* isometric embeddings of *compact* manifolds with *real analytic* Riemannian metrics to Euclidean spaces was proven by Nash in 1966.

[2]Nash proved this in his 1954 paper for $q \geq n + 2$, where he indicated that a modification of his method would allow $q = n + 1$ as well. This was implemented in a 1955 paper by Nico Kuiper.

the classical perspective, what Nash has achieved in his papers is as impossible as the story of his life.

Prior to Nash, the following two heuristic principles, vaguely similar to the first and the second laws of thermodynamics, have been (almost?) unquestionably accepted by analysts:

1. Conservation of Regularity. *The smoothness of solutions f of a "natural" functional, in particular a differential, equation $\mathscr{D}(f) = g$ is determined by the equation itself but not by a particular class of functions f used for the proof of the existence of solutions.*
2. Increase of Irregularity. *If some amount of regularity of potential solutions f of our equations has been lost, it cannot be recaptured by any "external means," such as artificial smoothing of functions.*

Instances of the first principle can be traced to the following three Hilbert's problems:

5th: *Continuous groups are infinitely differentiable, in fact, real analytic.*
19th: *Solutions of "natural" elliptic PDE are real analytic.*
Also Hilbert's formulation of his **13th** problem on

non-representability of "interesting" functions in many variables
by superpositions of ***continuous*** *functions in fewer variables*

is motivated by this principle:

$$\textit{continuous} \Leftrightarrow \textit{real analytic}$$

as far as superpositions of functions are concerned.

Nash C^1-isometric embedding theorem shattered the conservation of regularity idea: the system of differential equations that describes isometric immersions $f : X \to \mathbb{R}^q$ may have no analytic or not even C^2-smooth solution f.

But, according to Nash's 1954 theorem, *if $q > dim(X)$, and if X is diffeomorphic, to $\mathbb{R}^n$, $n < q$, or to the n-sphere, then, no matter what Riemannian metric g you are given on this X, there are lots of isometric C^1-embeddings $X \to \mathbb{R}^q$.*[3]

Now, look at an equally incredible Nash's approach to *more regular, say C^∞-smooth, isometric embeddings*. The main lemma used by Nash, his *implicit (or inverse) function theorem*, may seem "classical" unless you read the small print:

Let $\mathscr{D} : F \to G$ be a C^∞-smooth non-linear differential operator between spaces F and G of C^∞-sections of two vector bundles over a manifold X.

[3] In the spirit of Nash but probably independently, the *continuous* $\Leftrightarrow$ *real analytic* equivalence for superpositions of functions was disproved by Kolmogorov in 1956; yet, in essence, Hilbert's **13th** problem remains unsolved: *are there algebraic (or other natural) functions in many variables that are not superpositions of* ***real analytic*** *functions in two variables?*

Also, despite an enormous progress, "true" Hilbert's 19th problem remains widely open: *what are possible singularities of solutions of elliptic PDE systems, such as minimal subvarieties and Einstein manifolds.*

If the linearization $\mathscr{L} = \mathscr{L}_{f_0}(f)$ *of* $\mathscr{D}$ *at a point* $f_0 \in F$ *is invertible at* $g_0 = \mathscr{D}(f_0) \in G$ *by a* ***differential*** *operator linear in* g*, say* $\mathscr{M} = \mathscr{M}_{f_0}(g)$*, then* $\mathscr{D}$ *is also invertible (by a nonlinear non-differential operator) in a (small fine)* C^∞*-neighborhood of* $\mathscr{D}(f_0) \in G$.

You must be a novice in analysis or a genius like Nash to believe anything like that can be ever true and/or to have a single nontrivial application.

First of all, who has ever seen inversions of differential operators again by *differential* ones?

And second of all, how on earth can $\mathscr{D}$ be inverted by means of $\mathscr{M}$ when both operators, being differential, *increase* irregularity?

But Nash writes down a simple formula for a linearized inversion $\mathscr{M}$ for the metric inducing operator $\mathscr{D}$, and he suggests a compensation for the loss of regularity by the use of *smoothing operators*.

The latter may strike you as realistic as a successful performance of perpetuum mobile with a mechanical implementation of Maxwell's demon…unless you start following Nash's computation and realize to your immense surprise that the smoothing does work in the hands of John Nash.

This, combined with a few ingenious geometric constructions, leads to C^∞-smooth isometric embeddings $f : X \to \mathbb{R}^q$ for $q = 3n^3/2 + O(n^2)$, $n = dim(X)$.

Besides the above, Nash has proved a few other great theorems, but it is his work on isometric immersions that opened a new world of mathematics that stretches in front of our eyes in yet unknown directions and still waits to be explored.