

Parameterized Yang-Hilbert-Type Integral Inequalities and Their Operator Expressions

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Abstract Applying methods of Real Analysis and Functional Analysis, we build two weight functions with parameters and provide two kinds of parameterized Yang-Hilbert-type integral inequalities with the best constant factors. Equivalent forms, the reverses, and the operator expressions are also given. In particular, the Hardy-type inequalities and Hardy-type operators are studied. Additionally, a number of examples with two kinds of particular kernels are considered.

Key words Hardy-type integral operator; Yang-Hilbert-type integral inequality; Hölder's inequality; measurable function; weight function; equivalent form; operator expression

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1 Introduction

If $f(x), g(y) \geq 0$, satisfy

$$0 < \int_0^{\infty} f^2(x) dx < \infty$$

and

$$0 < \int_0^{\infty} g^2(y) dy < \infty,$$

then we have the following well known Hilbert's integral inequality (cf. [1])

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$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} dx dy < \pi \left(\int_0^\infty f^2(x) dx \int_0^\infty g^2(y) dy \right)^{\frac{1}{2}}, \quad (1)$$

where the constant factor π is the best possible. The operator expression of (1) was studied in [2] and [3].

In 1925, by introducing one pair of conjugate exponents (p, q) , that is $\frac{1}{p} + \frac{1}{q} = 1$, Hardy [4] provided an extension of (1) as follows:

For $p > 1$, $f(x), g(y) \geq 0$, satisfying

$$0 < \int_0^\infty f^p(x) dx < \infty$$

and

$$0 < \int_0^\infty g^q(y) dy < \infty,$$

we have

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} dx dy < \frac{\pi}{\sin(\pi/p)} \left(\int_0^\infty f^p(x) dx \right)^{\frac{1}{p}} \left(\int_0^\infty g^q(y) dy \right)^{\frac{1}{q}}, \quad (2)$$

where the constant factor $\frac{\pi}{\sin(\pi/p)}$ is still the best possible. Inequality (2) is known as Hardy-Hilbert's integral inequality, and is important in analysis and its applications (cf. [5], [6]).

Definition 1. If $\lambda \in \mathbf{R} = (-\infty, \infty)$, $k_\lambda(x, y)$ is a non-negative measurable function in $\mathbf{R}_+^2 = \mathbf{R}_+ \times \mathbf{R}_+$, satisfying

$$k_\lambda(tx, ty) = t^{-\lambda} k_\lambda(x, y),$$

for any $t, x, y > 0$, then we call $k_\lambda(x, y)$ the homogeneous function of degree $-\lambda$ in \mathbf{R}_+^2 .

In 1934, replacing $\frac{1}{x+y}$ in (2) by a general homogeneous kernel of degree -1, as $k_1(x, y)$, Hardy et al. presented an extension of (2) with the best possible constant factor

$$k_p = \int_0^\infty k_1(t, 1) t^{\frac{-1}{p}} dt \in \mathbf{R}_+ = (0, \infty)$$

obtaining (cf. [5], Th. 319):

$$\int_0^\infty \int_0^\infty k_1(x, y) f(x) g(y) dx dy < k_p \left(\int_0^\infty f^p(x) dx \right)^{\frac{1}{p}} \left(\int_0^\infty g^q(y) dy \right)^{\frac{1}{q}}. \quad (3)$$

The following inequality with the non-homogeneous kernel $h(xy)$ similar to (3) was studied (cf. [5], Th. 350):

For $h(t) > 0$, satisfying $\phi(s) := \int_0^\infty h(t) t^{s-1} dt \in \mathbf{R}_+$, $f(x), g(y) \geq 0$,

$$0 < \int_0^\infty x^{p-2} f^p(x) dx < \infty$$

and

$$0 < \int_0^{\infty} g^q(y)dy < \infty,$$

we have

$$\int_0^{\infty} \int_0^{\infty} h(xy)f(x)g(y)dxdy < \phi\left(\frac{1}{p}\right) \left(\int_0^{\infty} x^{p-2}f^p(x)dx\right)^{\frac{1}{p}} \left(\int_0^{\infty} g^q(y)dy\right)^{\frac{1}{q}}. \quad (4)$$

Remark 1. Hardy could not prove that the constant factor in (4) is the best possible and did not consider the operator expressions of (3) and (4) (cf. [5], Chapter 9). We shall call (3) and (4) Hardy-Hilbert-type integral inequalities, which only contain one pair of conjugate exponents (p, q) .

In 1998, by introducing an independent parameter $\lambda > 0$, Yang [7], [8] gave an extension of (1) as follows:

For $f(x), g(y) \geq 0$, such that

$$0 < \int_0^{\infty} x^{1-\lambda}f^2(x)dx < \infty$$

and

$$0 < \int_0^{\infty} y^{1-\lambda}g^2(y)dy < \infty,$$

we have

$$\int_0^{\infty} \int_0^{\infty} \frac{f(x)g(y)}{(x+y)^\lambda}dxdy < B\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right) \left(\int_0^{\infty} x^{1-\lambda}f^2(x)dx \int_0^{\infty} y^{1-\lambda}g^2(y)dy\right)^{\frac{1}{2}}, \quad (5)$$

where, the constant factor $B\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right)$ is the best possible, and

$$B(u, v) := \int_0^{\infty} \frac{t^{u-1}}{(t+1)^{u+v}}dt \quad (u, v > 0) \quad (6)$$

is the beta function (cf. [9]).

In 2004, by introducing two pairs of conjugate exponents (p, q) and (r, s) , that is $\frac{1}{p} + \frac{1}{q} = \frac{1}{r} + \frac{1}{s} = 1$, and an independent parameter $\lambda > 0$, Yang [10] gave the following extension of (3):

For $p, r > 1, f(x), g(y) \geq 0$, such that

$$0 < \int_0^{\infty} x^{p(1-\frac{\lambda}{r})-1}f^p(x)dx < \infty$$

and

$$0 < \int_0^{\infty} y^{q(1-\frac{\lambda}{s})-1}g^q(y)dy < \infty,$$

it holds

$$\begin{aligned} & \int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x^\lambda + y^\lambda} dx dy \\ & < \frac{\pi}{\lambda \sin(\pi/r)} \left[\int_0^\infty x^{p(1-\frac{\lambda}{r})-1} f^p(x) dx \right]^{\frac{1}{p}} \left[\int_0^\infty y^{q(1-\frac{\lambda}{s})-1} g^q(y) dy \right]^{\frac{1}{q}}, \quad (7) \end{aligned}$$

where, the constant factor

$$\frac{\pi}{\lambda \sin(\pi/r)}$$

is the best possible.

For $\lambda = 1, r = q, s = p$, inequality (7) reduces to (2); for $\lambda = 1, r = p, s = q$, inequality (7) reduces to the dual form of (2) as follows:

$$\begin{aligned} & \int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} dx dy \\ & < \frac{\pi}{\sin(\pi/p)} \left(\int_0^\infty x^{p-2} f^p(x) dx \right)^{\frac{1}{p}} \left(\int_0^\infty y^{q-2} g^q(y) dy \right)^{\frac{1}{q}}. \quad (8) \end{aligned}$$

In 2009, replacing $1/(x^\lambda + y^\lambda)$ in (7), by a general homogeneous kernel of degree $-\lambda$, as $k_\lambda(x, y)$, Yang [11], [12] proved an extension of (7) in the following form:

For $\lambda > 0, f(x), g(y) \geq 0$, such that

$$0 < \int_0^\infty x^{p(1-\frac{\lambda}{r})-1} f^p(x) dx < \infty$$

and

$$0 < \int_0^\infty y^{q(1-\frac{\lambda}{s})-1} g^q(y) dy < \infty,$$

it follows

$$\begin{aligned} & \int_0^\infty \int_0^\infty k_\lambda(x, y) f(x) g(y) dx dy \\ & < k_\lambda(r) \left[\int_0^\infty x^{p(1-\frac{\lambda}{r})-1} f^p(x) dx \right]^{\frac{1}{p}} \left[\int_0^\infty y^{q(1-\frac{\lambda}{s})-1} g^q(y) dy \right]^{\frac{1}{q}}, \quad (9) \end{aligned}$$

where, the constant factor

$$k_\lambda(r) := \int_0^\infty k_\lambda(t, 1) t^{\frac{\lambda}{r}-1} dt \in \mathbf{R}_+$$

is the best possible.

In [13], Yang presented also the following new inequality with a non-homogeneous kernel similar to (4):

For $f(x), g(y) \geq 0$, satisfying

$$0 < \int_0^\infty x^{p(1-\sigma)-1} f^p(x) dx < \infty$$

and

$$0 < \int_0^{\infty} y^{q(1-\sigma)-1} g^q(y) dy < \infty,$$

we have

$$\begin{aligned} & \int_0^{\infty} \int_0^{\infty} h(xy) f(x) g(y) dx dy \\ & < \phi(\sigma) \left(\int_0^{\infty} x^{p(1-\sigma)-1} f^p(x) dx \right)^{\frac{1}{p}} \left(\int_0^{\infty} y^{q(1-\sigma)-1} g^q(y) dy \right)^{\frac{1}{q}}, \end{aligned} \quad (10)$$

where, the constant factor

$$\phi(\sigma) = \int_0^{\infty} h(t) t^{\sigma-1} dt \in \mathbf{R}_+$$

is the best possible.

Remark 2. For $\lambda = 1, r = q, s = p$, it follows that inequality (9) reduces to (3). Hence, (9) is an extension of (3) with two pairs of conjugate exponents and an independent parameter. In 2014, Yang [14] proved that inequalities (9) and (10) are equivalent for

$$h(u) = k_{\lambda}(u, 1),$$

and considered the operator expressions of (9) and (10). We call (9) together with (10) as Yang-Hilbert-type integral inequalities in the first quadrant. Also we can call some similar inequalities as Yang-Hilbert-type integral inequalities in the whole plane.

In 2007, Yang [15] introduced a Hilbert-type integral inequality in the whole plane as follows:

For $f(x), g(y) \geq 0$, such that

$$0 < \int_{-\infty}^{\infty} e^{-\lambda x} f^2(x) dx < \infty$$

and

$$0 < \int_0^{\infty} \int_{-\infty}^{\infty} e^{-\lambda y} g^2(y) dy < \infty,$$

it holds

$$\begin{aligned} & \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{f(x)g(y)}{(1+e^{x+y})^{\lambda}} dx dy \\ & < B\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right) \left(\int_{-\infty}^{\infty} e^{-\lambda x} f^2(x) dx \int_{-\infty}^{\infty} e^{-\lambda y} g^2(y) dy \right)^{\frac{1}{2}}, \end{aligned} \quad (11)$$

where, the constant factor $B\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right)$ ($\lambda > 0$) is the best possible.

For the case when $0 < \lambda < 1, p > 1, \frac{1}{p} + \frac{1}{q} = 1$, Yang [16] proved in 2008, the following Hilbert-type integral inequality in the whole plane:

For $p > 1$, $f(x), g(y) \geq 0$, satisfying

$$0 < \int_{-\infty}^{\infty} |x|^{p(1-\frac{\lambda}{2})-1} f^p(x) dx < \infty$$

and

$$0 < \int_{-\infty}^{\infty} |y|^{q(1-\frac{\lambda}{2})-1} g^q(y) dy < \infty,$$

we have

$$\begin{aligned} & \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{|1+xy|^\lambda} f(x)g(y) dx dy \\ & < k_\lambda \left[\int_{-\infty}^{\infty} |x|^{p(1-\frac{\lambda}{2})-1} f^p(x) dx \right]^{\frac{1}{p}} \left[\int_{-\infty}^{\infty} |y|^{q(1-\frac{\lambda}{2})-1} g^q(y) dy \right]^{\frac{1}{q}}, \end{aligned} \quad (12)$$

where, the constant

$$k_\lambda = B\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right) + 2B\left(1-\lambda, \frac{\lambda}{2}\right)$$

is still the best possible.

Additionally, Yang et al. [17]-[26] provided also some other Hilbert-type integral inequalities in the whole plane. Rassias, M. Th. et al. [27]-[32] presented as well some different new Hilbert-type inequalities.

In this paper, applying methods of Real Analysis and Functional Analysis, we build two weight functions with parameters, and provide two kinds of parameterized Yang-Hilbert-type integral inequalities with the best constant factors. Equivalent forms, the reverses, and the operator expressions are also given. In particular, the Hardy-type inequalities and Hardy-type operators are studied. Furthermore, a number of examples with two kinds of particular kernels are considered.

2 Yang-Hilbert-Type Integral Inequalities in the First Quadrant

In this section, we present a weight function and study some Yang-Hilbert-type integral inequalities in the first quadrant with parameters and the best constant factors. Equivalent forms, the reverses, the Hardy-type inequalities, the operator expressions and some particular examples are also discussed.

2.1 Definition of Weight Function and a Lemma

Definition 2. If $\sigma \in \mathbf{R}$, $h(t)$ is a non-negative measurable function in \mathbf{R}_+ , define the following weight function:

$$\omega(\sigma, y) := y^\sigma \int_0^\infty h(xy)x^{\sigma-1} dx (y \in \mathbf{R}_+). \quad (13)$$

Setting $t = xy$ in (13), we obtain

$$\omega(\sigma, y) = k(\sigma) := \int_0^\infty h(t)t^{\sigma-1} dt. \quad (14)$$

Lemma 1. *If $p > 0$ ($p \neq 1$), $\frac{1}{p} + \frac{1}{q} = 1$, $\sigma \in \mathbf{R}$, both $h(t)$ and $f(t)$ are non-negative measurable functions in \mathbf{R}_+ , and $k(\sigma)$ is defined by (14), then, (i) for $p > 1$, we have the following inequality:*

$$J := \int_0^\infty y^{p\sigma-1} \left(\int_0^\infty h(xy)f(x)dx \right)^p dy \leq k^p(\sigma) \int_0^\infty x^{p(1-\sigma)-1} f^p(x)dx; \quad (15)$$

(ii) for $0 < p < 1$, we have the reverse of (15).

Proof. (i) By the weighted Hölder's inequality (cf. [33]) and (13), it follows that

$$\begin{aligned} \int_0^\infty h(xy)f(x)dx &= \int_0^\infty h(xy) \left[\frac{x^{(1-\sigma)/q}}{y^{(1-\sigma)/p}} f(x) \right] \left[\frac{y^{(1-\sigma)/p}}{x^{(1-\sigma)/q}} \right] dx \\ &\leq \left[\int_0^\infty h(xy) \frac{x^{(1-\sigma)p/q}}{y^{1-\sigma}} f^p(x)dx \right]^{\frac{1}{p}} \left[\int_0^\infty h(xy) \frac{y^{(1-\sigma)q/p}}{x^{1-\sigma}} dx \right]^{\frac{1}{q}} \\ &= (\omega(\sigma, y))^{\frac{1}{q}} y^{\frac{1}{p}-\sigma} \left[\int_0^\infty h(xy) \frac{x^{(1-\sigma)(p-1)}}{y^{1-\sigma}} f^p(x)dx \right]^{\frac{1}{p}}. \end{aligned} \quad (16)$$

Then by (14) and Fubini's theorem (cf. [34]), we have

$$\begin{aligned} J &\leq k^{p-1}(\sigma) \int_0^\infty \int_0^\infty h(xy) \frac{x^{(1-\sigma)(p-1)}}{y^{1-\sigma}} f^p(x) dx dy \\ &= k^{p-1}(\sigma) \int_0^\infty \left[\int_0^\infty h(xy) \frac{x^{(1-\sigma)(p-1)}}{y^{1-\sigma}} dy \right] f^p(x) dx \\ &= k^{p-1}(\sigma) \int_0^\infty \omega(\sigma, x) x^{p(1-\sigma)-1} f^p(x) dx. \end{aligned} \quad (17)$$

By (14), we obtain (15).

(ii) For $0 < p < 1$, by the reverse of the weighted Hölder's inequality (cf. [33]), combined with (13) and (14), we obtain the reverse of (16) and (17). Then we get the reverse of (15) by using (14). This completes the proof of the lemma.

2.2 Two Equivalent Inequalities as well as the Reverses with the Best Possible Constant Factors

Theorem 1. Suppose that $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $\sigma \in \mathbf{R}$, $h(t) \geq 0$, and

$$k(\sigma) = \int_0^{\infty} h(t)t^{\sigma-1} dt \in \mathbf{R}_+.$$

If $f(x), g(y) \geq 0$, such that

$$0 < \int_0^{\infty} x^{p(1-\sigma)-1} f^p(x) dx < \infty$$

and

$$0 < \int_0^{\infty} y^{q(1-\sigma)-1} g^q(y) dy < \infty,$$

then we have the following equivalent inequalities:

$$\begin{aligned} I &:= \int_0^{\infty} \int_0^{\infty} h(xy) f(x) g(y) dx dy \\ &< k(\sigma) \left[\int_0^{\infty} x^{p(1-\sigma)-1} f^p(x) dx \right]^{\frac{1}{p}} \left[\int_0^{\infty} y^{q(1-\sigma)-1} g^q(y) dy \right]^{\frac{1}{q}}, \end{aligned} \quad (18)$$

$$J = \int_0^{\infty} y^{p\sigma-1} \left(\int_0^{\infty} h(xy) f(x) dx \right)^p dy < k^p(\sigma) \int_0^{\infty} x^{p(1-\sigma)-1} f^p(x) dx, \quad (19)$$

where, the constant factors $k(\sigma)$ and $k^p(\sigma)$ are the best possible.

Proof. We first proved that (16) preserves the form of a strict inequality for any $y \in \mathbf{R}_+$. Otherwise, there exists a $y > 0$, such that (16) becomes an equality. Then, there exist two constants A and B , such that they are not all zero, and (cf. [33])

$$A \frac{x^{(1-\sigma)p/q}}{y^{1-\sigma}} f^p(x) = B \frac{y^{(1-\sigma)q/p}}{x^{1-\sigma}} \quad \text{a. e. in } \mathbf{R}_+.$$

If $A = 0$, then $B = 0$, which is impossible. Suppose that $A \neq 0$. Then it follows that

$$x^{p(1-\sigma)-1} f^p(x) = y^{(1-\sigma)q} \frac{B}{Ax} \quad \text{a. e. in } \mathbf{R}_+,$$

which contradicts the fact that

$$0 < \int_0^{\infty} x^{p(1-\sigma)-1} f^p(x) dx < \infty,$$

in virtue of

$$\int_0^{\infty} \frac{1}{x} dx = \infty.$$

Hence, both (16) and (17) preserve the forms of strict inequalities, and thus we have (19).

By Hölder's inequality (cf. [33]), we obtain

$$\begin{aligned} I &= \int_0^\infty \left(y^{\sigma-\frac{1}{p}} \int_0^\infty h(xy) f(x) dx \right) (y^{\frac{1}{p}-\sigma} g(y)) dy \\ &\leq J^{\frac{1}{p}} \left[\int_0^\infty y^{q(1-\sigma)-1} g^q(y) dy \right]^{\frac{1}{q}}. \end{aligned} \quad (20)$$

Then by (19), we get (18). On the other hand, assuming that (18) is valid, we set

$$g(y) := y^{p\sigma-1} \left(\int_0^\infty h(xy) f(x) dx \right)^{p-1}, y \in \mathbf{R}_+.$$

Then we obtain

$$J = \int_0^\infty y^{q(1-\sigma)-1} g^q(y) dy.$$

By (15), in view of

$$0 < \int_0^\infty x^{p(1-\sigma)-1} f^p(x) dx < \infty,$$

it follows that $J < \infty$. If $J = 0$, then, (19) is trivially valid; if $J > 0$, then by (18), we have

$$\begin{aligned} 0 < \int_0^\infty y^{q(1-\sigma)-1} g^q(y) dy &= J = I \\ &< k(\sigma) \left[\int_0^\infty x^{p(1-\sigma)-1} f^p(x) dx \right]^{\frac{1}{p}} \left[\int_0^\infty y^{q(1-\sigma)-1} g^q(y) dy \right]^{\frac{1}{q}}, \end{aligned}$$

$$J^{\frac{1}{p}} = \left[\int_0^\infty y^{q(1-\sigma)-1} g^q(y) dy \right]^{\frac{1}{p}} < k(\sigma) \left[\int_0^\infty x^{p(1-\sigma)-1} f^p(x) dx \right]^{\frac{1}{p}},$$

and then (19) follows, which is equivalent to (18).

For any $n \in \mathbf{N}$ (where \mathbf{N} is the set of positive integers), we define the functions $f_n(x)$ and $g_n(y)$ as follows:

$$f_n(x) := \begin{cases} 0, & x \in (0, 1) \\ x^{\sigma-\frac{1}{np}-1}, & x \in [1, \infty) \end{cases}, \quad g_n(y) := \begin{cases} y^{\sigma+\frac{1}{nq}-1}, & y \in (0, 1] \\ 0, & y \in (1, \infty) \end{cases}.$$

Then we find

$$L_n := \left[\int_0^\infty x^{p(1-\sigma)-1} f_n^p(x) dx \right]^{\frac{1}{p}} \left[\int_0^\infty y^{q(1-\sigma)-1} g_n^q(y) dy \right]^{\frac{1}{q}}$$

$$= \left(\int_1^\infty x^{-\frac{1}{n}-1} dx \right)^{\frac{1}{p}} \left(\int_0^1 y^{\frac{1}{n}-1} dy \right)^{\frac{1}{q}} = n.$$

By Fubini's theorem, it follows that

$$\begin{aligned} I_n &:= \int_0^\infty \int_0^\infty h(xy) f_n(x) g_n(y) dx dy \\ &= \int_1^\infty x^{\sigma - \frac{1}{np} - 1} \left(\int_0^1 h(xy) y^{\sigma + \frac{1}{nq} - 1} dy \right) dx \\ &= \int_1^\infty x^{-\frac{1}{n} - 1} \left(\int_0^x h(t) t^{\sigma + \frac{1}{nq} - 1} dt \right) dx \\ &= \int_1^\infty x^{-\frac{1}{n} - 1} \left(\int_0^1 h(t) t^{\sigma + \frac{1}{nq} - 1} dt + \int_1^x h(t) t^{\sigma + \frac{1}{nq} - 1} dt \right) dx \\ &= n \int_0^1 h(t) t^{\sigma + \frac{1}{nq} - 1} dt + \int_1^\infty x^{-\frac{1}{n} - 1} \left(\int_1^x h(t) t^{\sigma + \frac{1}{nq} - 1} dt \right) dx \\ &= n \int_0^1 h(t) t^{\sigma + \frac{1}{nq} - 1} dt + \int_1^\infty \left(\int_t^\infty x^{-\frac{1}{n} - 1} dx \right) h(t) t^{\sigma + \frac{1}{nq} - 1} dt \\ &= n \left(\int_0^1 h(t) t^{\sigma + \frac{1}{nq} - 1} dt + \int_1^\infty h(t) t^{\sigma - \frac{1}{np} - 1} dt \right). \end{aligned}$$

If there exists a positive number $k \leq k(\sigma)$, such that (18) is still valid when replacing $k(\sigma)$ by k , then in particular, it follows that

$$\frac{1}{n} I_n < k \frac{1}{n} L_n,$$

and

$$\int_0^1 h(t) t^{\sigma + \frac{1}{nq} - 1} dt + \int_1^\infty h(t) t^{\sigma - \frac{1}{np} - 1} dt < k.$$

Since both

$$\{h(t) t^{\sigma + \frac{1}{nq} - 1}\}_{n=1}^\infty \quad (t \in (0, 1])$$

and

$$\{h(t) t^{\sigma - \frac{1}{np} - 1}\}_{n=1}^\infty \quad (t \in (1, \infty))$$

are non-negative and increasing, then by Levi's theorem (cf. [34]), we get

$$\begin{aligned} k(\sigma) &= \int_0^1 h(t) t^{\sigma - 1} dt + \int_1^\infty h(t) t^{\sigma - 1} dt \\ &= \lim_{n \rightarrow \infty} \left(\int_0^1 h(t) t^{\sigma + \frac{1}{nq} - 1} dt + \int_1^\infty h(t) t^{\sigma - \frac{1}{np} - 1} dt \right) \leq k. \end{aligned}$$

Thus $k = k(\sigma)$ is the best possible constant factor of (18).

The constant factor in (19) is still the best possible. Otherwise, by (20) we would reach the contradiction that the constant factor in (18) is not the best possible. This completes the proof of the theorem.

Theorem 2. Replacing $p > 1$ by $0 < p < 1$ in Theorem 1, we obtain the equivalent reverses of (18) and (19). If there exists a constant $\delta_0 > 0$, such that for any $\tilde{\sigma} \in (\sigma - \delta_0, \sigma]$,

$$k(\tilde{\sigma}) = \int_0^{\infty} h(t)t^{\tilde{\sigma}-1} dt \in \mathbf{R}_+,$$

then the constant factors in the reverses of (18) and (19) are the best possible.

Proof. By Lemma 1 and the reverse of Hölder's inequality, we get the reverses of (18), (19) and (20). Similarly, we can set $g(y)$ as in Theorem 1, and prove that the reverses of (18) and (19) are equivalent.

For $n > \frac{2}{\delta_0|q|}$ ($n \in \mathbf{N}$), we set $f_n(x)$ and $g_n(y)$ as in Theorem 1. If there exists a positive number $k \geq k(\sigma)$, such that the reverse of (18) is valid when replacing $k(\sigma)$ by k , then it follows that

$$\frac{1}{n}I_n > k\frac{1}{n}L_n,$$

and

$$\int_0^1 h(t)t^{\sigma+\frac{1}{nq}-1} dt + \int_1^{\infty} h(t)t^{\sigma-\frac{1}{np}-1} dt > k. \quad (21)$$

Since $\{h(t)t^{\sigma-\frac{1}{np}-1}\}_{n=1}^{\infty}$ ($t \in (1, \infty)$) is still non-negative and increasing, by Levi's theorem it follows that

$$\lim_{n \rightarrow \infty} \int_1^{\infty} h(t)t^{\sigma-\frac{1}{np}-1} dt = \int_1^{\infty} h(t)t^{\sigma-1} dt.$$

Since

$$0 \leq h(t)t^{\sigma+\frac{1}{nq}-1} \leq h(t)t^{(\sigma-\frac{\delta_0}{2})-1} \quad \left(t \in (0, 1], n > \frac{2}{\delta_0|q|} \right),$$

and

$$0 \leq \int_0^1 h(t)t^{(\sigma-\frac{\delta_0}{2})-1} dt \leq k(\sigma - \frac{\delta_0}{2}) < \infty,$$

then by Lebesgue's dominated convergence theorem (cf. [34]), it follows that

$$\lim_{n \rightarrow \infty} \int_0^1 h(t)t^{\sigma+\frac{1}{nq}-1} dt = \int_0^1 h(t)t^{\sigma-1} dt.$$

In view of the above results and (21), we have

$$k(\sigma) = \lim_{n \rightarrow \infty} \left(\int_0^1 h(t)t^{\sigma+\frac{1}{nq}-1} dt + \int_1^{\infty} h(t)t^{\sigma-\frac{1}{np}-1} dt \right) \geq k.$$

Then $k = k(\sigma)$ is the best possible constant factor for the reverse of (18).

Following the same method, we can prove that the constant factor in the reverse of (19) is the best possible, by the use of the reverse of (20). This completes the proof of the theorem.

2.3 Yang-Hilbert-Type Integral Inequalities in the First Quadrant with Multi-Variables

Theorem 3. Suppose that $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $h(t) \geq 0$, $\sigma \in \mathbf{R}$,

$$k(\sigma) = \int_0^\infty h(t)t^{\sigma-1} dt \in \mathbf{R}_+,$$

$\delta_i \in \{-1, 1\}$, $0 \leq a_i < b_i \leq \infty$, $v_i'(s) > 0$ ($s \in (a_i, b_i)$) and

$$v_i(a_i^+) = \lim_{s \rightarrow a_i^+} v_i(s) = 0,$$

$$v_i(b_i^-) = \lim_{s \rightarrow b_i^-} v_i(s) = \infty \quad (i = 1, 2).$$

If $f(x), g(y) \geq 0$, such that

$$0 < \int_{a_1}^{b_1} \frac{(v_1(x))^{p(1-\delta_1\sigma)-1}}{(v_1'(x))^{p-1}} f^p(x) dx < \infty$$

and

$$0 < \int_{a_2}^{b_2} \frac{(v_2(y))^{q(1-\delta_2\sigma)-1}}{(v_2'(y))^{q-1}} g^q(y) dy < \infty,$$

then we have the following equivalent inequalities:

$$\begin{aligned} & \int_{a_2}^{b_2} \int_{a_1}^{b_1} h(v_1^{\delta_1}(x)v_2^{\delta_2}(y))f(x)g(y)dxdy \\ & < k(\sigma) \left[\int_{a_1}^{b_1} \frac{(v_1(x))^{p(1-\delta_1\sigma)-1}}{(v_1'(x))^{p-1}} f^p(x) dx \right]^{\frac{1}{p}} \\ & \quad \times \left[\int_{a_2}^{b_2} \frac{(v_2(y))^{q(1-\delta_2\sigma)-1}}{(v_2'(y))^{q-1}} g^q(y) dy \right]^{\frac{1}{q}}, \end{aligned} \quad (22)$$

$$\begin{aligned} & \int_{a_2}^{b_2} \frac{v_2'(y)}{(v_2(y))^{1-p\delta_2\sigma}} \left(\int_{a_1}^{b_1} h(v_1^{\delta_1}(x)v_2^{\delta_2}(y))f(x) dx \right)^p dy \\ & < k^p(\sigma) \int_{a_1}^{b_1} \frac{(v_1(x))^{p(1-\delta_1\sigma)-1}}{(v_1'(x))^{p-1}} f^p(x) dx, \end{aligned} \quad (23)$$

where the constant factors $k(\sigma)$ and $k^p(\sigma)$ are the best possible.

Proof. Setting

$$x = v_1^{\delta_1}(s), y = v_2^{\delta_2}(t)$$

in (18), since $\delta_i \in \{-1, 1\}$, we get

$$dx = \delta_1 v_1^{\delta_1-1}(s) v_1'(s) ds, \quad dy = \delta_2 v_2^{\delta_2-1}(t) v_2'(t) dt,$$

and

$$\begin{aligned} I &= |\delta_1 \delta_2| \int_{a_2}^{b_2} \int_{a_1}^{b_1} h(v_1^{\delta_1}(s) v_2^{\delta_2}(t)) f(v_1^{\delta_1}(s)) g(v_2^{\delta_2}(t)) v_1^{\delta_1-1}(s) v_1'(s) v_2^{\delta_2-1}(t) v_2'(t) ds dt \\ &= \int_{a_2}^{b_2} \int_{a_1}^{b_1} h(v_1^{\delta_1}(s) v_2^{\delta_2}(t)) (f(v_1^{\delta_1}(s)) v_1^{\delta_1-1}(s) v_1'(s)) (g(v_2^{\delta_2}(t)) v_2^{\delta_2-1}(t) v_2'(t)) ds dt, \end{aligned}$$

$$I_1 := \int_0^\infty x^{p(1-\sigma)-1} f^p(x) dx = \int_{a_1}^{b_1} (v_1^{\delta_1}(s))^{p(1-\sigma)-1} f^p(v_1^{\delta_1}(s)) v_1^{\delta_1-1}(s) v_1'(s) ds,$$

$$I_2 := \int_0^\infty y^{q(1-\sigma)-1} g^q(y) dy = \int_{a_2}^{b_2} (v_2^{\delta_2}(t))^{q(1-\sigma)-1} g^q(v_2^{\delta_2}(t)) v_2^{\delta_2-1}(t) v_2'(t) dt.$$

Setting

$$F(s) = f(v_1^{\delta_1}(s)) v_1^{\delta_1-1}(s) v_1'(s), \quad G(t) = g(v_2^{\delta_2}(t)) v_2^{\delta_2-1}(t) v_2'(t),$$

we obtain

$$\begin{aligned} f^p(v_1^{\delta_1}(s)) &= v_1^{p(1-\delta_1)}(s) (v_1'(s))^{-p} F^p(s), \\ g^q(v_2^{\delta_2}(t)) &= v_2^{q(1-\delta_2)}(t) (v_2'(t))^{-q} G^q(t), \end{aligned}$$

and then it follows that

$$\begin{aligned} I &= \int_{a_2}^{b_2} \int_{a_1}^{b_1} h(v_1^{\delta_1}(s) v_2^{\delta_2}(t)) F(s) G(t) ds dt, \\ I_1 &= \int_{a_1}^{b_1} \frac{(v_1(s))^{p(1-\delta_1\sigma)-1}}{(v_1'(s))^{p-1}} F^p(s) ds, \quad I_2 = \int_{a_2}^{b_2} \frac{(v_2(t))^{q(1-\delta_2\sigma)-1}}{(v_2'(t))^{q-1}} G^q(t) dt. \end{aligned}$$

Substituting the above results in (18), resetting

$$s = x, t = y, F(s) = f(x), G(t) = g(y),$$

we obtain (22). Similarly, we have (23).

On the other hand, if we set

$$v_1^{\delta_1}(x) = x, v_2^{\delta_2}(y) = y, a_i = 0, b_i = \infty \quad (i = 1, 2)$$

in (22), we obtain (18). Hence, the inequalities (22) and (18) are equivalent. It is evident that the inequalities (23) and (19) are equivalent. Hence, the inequalities (22) and (23) are equivalent. Since the constant factors in (18) and (19) are the best possible, it follows that the constant factors in (22) and (23) are also the best possible by using the equivalency.

This completes the proof of the theorem.

Theorem 4. *Replacing $p > 1$ by $0 < p < 1$ in Theorem 3, we obtain the equivalent reverses of (22) and (23). If there exists a constant $\delta_0 > 0$, such that for any value of $\tilde{\sigma} \in (\sigma - \delta_0, \sigma]$,*

$$k(\tilde{\sigma}) = \int_0^\infty h(t)t^{\tilde{\sigma}-1} dt \in \mathbf{R}_+,$$

then the constant factors in the reverses of (22) and (23) are the best possible.

In particular, for $\delta_1 = \delta_2 = 1$ in Theorem 3 and Theorem 4, we get the following integral inequalities with the non-homogeneous kernel:

Corollary 1. *Suppose that $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $h(t) \geq 0$, $\sigma \in \mathbf{R}$,*

$$k(\sigma) = \int_0^\infty h(t)t^{\sigma-1} dt \in \mathbf{R}_+,$$

$0 \leq a_i < b_i \leq \infty$, $v_i'(s) > 0$ ($s \in (a_i, b_i)$), $v_i(a_i^+) = 0$, $v_i(b_i^-) = \infty$ ($i = 1, 2$). If $f(x), g(y) \geq 0$, such that

$$0 < \int_{a_1}^{b_1} \frac{(v_1(x))^{p(1-\sigma)-1}}{(v_1'(x))^{p-1}} f^p(x) dx < \infty$$

and

$$0 < \int_{a_2}^{b_2} \frac{(v_2(y))^{q(1-\sigma)-1}}{(v_2'(y))^{q-1}} g^q(y) dy < \infty,$$

then we have the following equivalent inequalities:

$$\begin{aligned} & \int_{a_2}^{b_2} \int_{a_1}^{b_1} h(v_1(x)v_2(y)) f(x)g(y) dx dy \\ & < k(\sigma) \left[\int_{a_1}^{b_1} \frac{(v_1(x))^{p(1-\sigma)-1}}{(v_1'(x))^{p-1}} f^p(x) dx \right]^{\frac{1}{p}} \\ & \quad \times \left[\int_{a_2}^{b_2} \frac{(v_2(y))^{q(1-\sigma)-1}}{(v_2'(y))^{q-1}} g^q(y) dy \right]^{\frac{1}{q}}, \end{aligned} \quad (24)$$

$$\begin{aligned} & \int_{a_2}^{b_2} \frac{v_2'(y)}{(v_2(y))^{1-p\sigma}} \left(\int_{a_1}^{b_1} h(v_1(x)v_2(y)) f(x) dx \right)^p dy \\ & < k^p(\sigma) \int_{a_1}^{b_1} \frac{(v_1(x))^{p(1-\sigma)-1}}{(v_1'(x))^{p-1}} f^p(x) dx, \end{aligned} \quad (25)$$

where the constant factors $k(\sigma)$ and $k^p(\sigma)$ are the best possible.

Replacing $p > 1$ by $0 < p < 1$ in the above inequalities, we obtain the equivalent reverses of (24) and (25).

If there exists a constant $\delta_0 > 0$, such that for any $\tilde{\sigma} \in (\sigma - \delta_0, \sigma]$,

$$k(\tilde{\sigma}) = \int_0^\infty h(t)t^{\tilde{\sigma}-1} dt \in \mathbf{R}_+,$$

then the constant factors in the reverses of (24) and (25) are the best possible.

In particular, for $\delta_1 = -1, \delta_2 = 1$ in Theorem 3 and Theorem 4, setting

$$h(t) = k_\lambda(1, t)$$

(cf. Definition 1), we find

$$h\left(\frac{v_2(y)}{v_1(x)}\right) = k_\lambda\left(1, \frac{v_2(y)}{v_1(x)}\right) = v_1^\lambda(x)k_\lambda(v_1(x), v_2(y)).$$

Replacing $f(x)$ by $v_1^{-\lambda}(x)f(x)$, it follows that $[v_1(x)]^{p(1+\sigma)-1}f^p(x)$ is replaced by

$$[v_1(x)]^{p(1+\sigma)-1}[v_1^{-\lambda}(x)f(x)]^p = [v_1(x)]^{p(1-\mu)-1}f^p(x),$$

and we have the following integral inequalities with the homogeneous kernel:

Corollary 2. Suppose that $p > 1, \frac{1}{p} + \frac{1}{q} = 1, \mu, \sigma \in \mathbf{R}, \mu + \sigma = \lambda, k_\lambda(x, y)$ is a homogeneous function of degree $-\lambda$ in \mathbf{R}_+^2 ,

$$k_\lambda(\sigma) = \int_0^\infty k_\lambda(1, t)t^{\sigma-1} dt \in \mathbf{R}_+,$$

$0 \leq a_i < b_i \leq \infty, v_i'(s) > 0$ ($s \in (a_i, b_i)$), $v_i(a_i^+) = 0, v_i(b_i^-) = \infty$ ($i = 1, 2$). If $f(x), g(y) \geq 0$, such that

$$0 < \int_{a_1}^{b_1} \frac{(v_1(x))^{p(1-\mu)-1}}{(v_1'(x))^{p-1}} f^p(x) dx < \infty$$

and

$$0 < \int_{a_2}^{b_2} \frac{(v_2(y))^{q(1-\sigma)-1}}{(v_2'(y))^{q-1}} g^q(y) dy < \infty,$$

then we have the following equivalent inequalities:

$$\begin{aligned} & \int_{a_2}^{b_2} \int_{a_1}^{b_1} k_\lambda(v_1(x), v_2(y)) f(x) g(y) dx dy \\ & < k_\lambda(\sigma) \left[\int_{a_1}^{b_1} \frac{(v_1(x))^{p(1-\mu)-1}}{(v_1'(x))^{p-1}} f^p(x) dx \right]^{\frac{1}{p}} \end{aligned}$$

$$\times \left[\int_{a_2}^{b_2} \frac{(v_2(y))^{q(1-\sigma)-1}}{(v_2'(y))^{q-1}} g^q(y) dy \right]^{\frac{1}{q}}, \quad (26)$$

$$\begin{aligned} & \int_{a_2}^{b_2} \frac{v_2'(y)}{(v_2(y))^{1-p\sigma}} \left(\int_{a_1}^{b_1} k_\lambda(v_1(x), v_2(y)) f(x) dx \right)^p dy \\ & < k_\lambda^p(\sigma) \int_{a_1}^{b_1} \frac{(v_1(x))^{p(1-\mu)-1}}{(v_1'(x))^{p-1}} f^p(x) dx, \end{aligned} \quad (27)$$

where the constant factors $k_\lambda(\sigma)$ and $k_\lambda^p(\sigma)$ are the best possible.

Replacing $p > 1$ by $0 < p < 1$ in the above inequalities, we obtain the equivalent reverses of (26) and (27). If there exists a constant $\delta_0 > 0$, such that for any $\tilde{\sigma} \in (\sigma - \delta_0, \sigma]$,

$$k_\lambda(\tilde{\sigma}) = \int_0^\infty k_\lambda(1, t) t^{\tilde{\sigma}-1} dt \in \mathbf{R}_+,$$

then the constant factors in the reverses of (26) and (27) are the best possible.

Setting

$$a_i = 0, b_i = \infty \quad (i = 1, 2), v_1(x) = x, v_2(y) = y$$

in Corollary 2, we have

Corollary 3. Suppose that $p > 1, \frac{1}{p} + \frac{1}{q} = 1, \mu, \sigma \in \mathbf{R}, \mu + \sigma = \lambda, k_\lambda(x, y)$ is a homogeneous function of degree $-\lambda$ in \mathbf{R}_+^2 , and

$$k_\lambda(\sigma) = \int_0^\infty k_\lambda(1, t) t^{\sigma-1} dt \in \mathbf{R}_+.$$

If $f(x), g(y) \geq 0$,

$$0 < \int_0^\infty x^{p(1-\mu)-1} f^p(x) dx < \infty$$

and

$$0 < \int_0^\infty y^{q(1-\sigma)-1} g^q(y) dy < \infty,$$

then we have the following equivalent inequalities:

$$\begin{aligned} & \int_0^\infty \int_0^\infty k_\lambda(x, y) f(x) g(y) dx dy \\ & < k_\lambda(\sigma) \left[\int_0^\infty x^{p(1-\mu)-1} f^p(x) dx \right]^{\frac{1}{p}} \left[\int_0^\infty y^{q(1-\sigma)-1} g^q(y) dy \right]^{\frac{1}{q}}, \end{aligned} \quad (28)$$

$$\int_0^\infty y^{p\sigma-1} \left(\int_0^\infty k_\lambda(x, y) f(x) dx \right)^p dy < k_\lambda^p(\sigma) \int_0^\infty x^{p(1-\mu)-1} f^p(x) dx, \quad (29)$$

where, the constant factors $k_\lambda(\sigma)$ and $k_\lambda^p(\sigma)$ are the best possible.

Replacing $p > 1$ by $0 < p < 1$ in the above inequalities, we obtain the equivalent reverses of (28) and (29). If there exists a constant $\delta_0 > 0$, such that for any $\tilde{\sigma} \in (\sigma - \delta_0, \sigma]$,

$$k_\lambda(\tilde{\sigma}) = \int_0^\infty k_\lambda(1, t)t^{\tilde{\sigma}-1} dt \in \mathbf{R}_+,$$

then the constant factors in the reverses of (28) and (29) are the best possible.

Remark 3. (a) It is evident that (18) and (28) are equivalent for $h(t) = k_\lambda(1, t)$. The same holds for (19) and (29).

(b) In the following, we list the functions $v_i(s)$ ($i = 1, 2$) which satisfy the conditions of Theorem 3 and Theorem 4:

- (i) $v_i(s) = s^a, s \in (0, \infty)$ ($a \in \mathbf{R}_+$), with $v'_i(s) = as^{a-1} > 0$;
- (ii) $v_i(s) = \tan^a s, s \in (0, \frac{\pi}{2})$ ($a \in \mathbf{R}_+$), with $v'_i(s) = a \tan^{a-1} s \sec^2 s > 0$;
- (iii) $v_i(s) = \ln^a s, s \in (1, \infty)$ ($a \in \mathbf{R}_+$), with $v'_i(s) = \frac{a}{s} \ln^{a-1} s > 0$;
- (iv) $v_i(s) = e^{as} - 1, s \in (0, \infty)$ ($a \in \mathbf{R}_+$), with $v'_i(s) = ae^{as} > 0$.

2.4 Hardy-Type Integral Inequalities with Multi-Variables

In the following two sections, if the constant factors in the inequalities (operator inequalities) are related to $k^{(1)}(\sigma)$ (or $k_\lambda^{(1)}(\sigma)$), then we call them Hardy-type inequalities (operators) of the *first* kind; if the constant factors in the inequalities (operator inequalities) are related to $k^{(2)}(\sigma)$ (or $k_\lambda^{(2)}(\sigma)$), then we call them Hardy-type inequalities (operators) of the *second* kind.

If $h(t) = 0$ ($t > 1$), then we have $h(xy) = 0$ ($x > \frac{1}{y} > 0$), and

$$k(\sigma) = \int_0^\infty h(t)t^{\sigma-1} dt = \int_0^1 h(t)t^{\sigma-1} dt.$$

Setting

$$k^{(1)}(\sigma) := \int_0^1 h(t)t^{\sigma-1} dt, \tag{30}$$

by Theorem 1 and Theorem 2, we obtain the following first kind Hardy-type integral inequalities with non-homogeneous kernel:

Corollary 4. Suppose that $p > 1, \frac{1}{p} + \frac{1}{q} = 1, h(t) \geq 0, \sigma \in \mathbf{R}$,

$$k^{(1)}(\sigma) = \int_0^1 h(t)t^{\sigma-1} dt \in \mathbf{R}_+.$$

If $f(x), g(y) \geq 0$,

$$0 < \int_0^\infty x^{p(1-\sigma)-1} f^p(x) dx < \infty$$

and

$$0 < \int_0^{\infty} y^{q(1-\sigma)-1} g^q(y) dy < \infty,$$

then we have the following equivalent inequalities:

$$\begin{aligned} & \int_0^{\infty} \left(\int_0^{\frac{1}{y}} h(xy) f(x) dx \right) g(y) dy = \int_0^{\infty} \left(\int_0^{\frac{1}{x}} h(xy) g(y) dy \right) f(x) dx \\ & < k^{(1)}(\sigma) \left[\int_0^{\infty} x^{p(1-\sigma)-1} f^p(x) dx \right]^{\frac{1}{p}} \left[\int_0^{\infty} y^{q(1-\sigma)-1} g^q(y) dy \right]^{\frac{1}{q}}, \end{aligned} \quad (31)$$

$$\int_0^{\infty} y^{p\sigma-1} \left(\int_0^{\frac{1}{y}} h(xy) f(x) dx \right)^p dy < (k^{(1)}(\sigma))^p \int_0^{\infty} x^{p(1-\sigma)-1} f^p(x) dx; \quad (32)$$

where, the constant factors $k^{(1)}(\sigma)$ and $(k^{(1)}(\sigma))^p$ are the best possible.

Replacing $p > 1$ by $0 < p < 1$ in the above inequalities, we obtain the equivalent reverses of (31) and (32). If there exists a constant $\delta_0 > 0$, such that for any $\tilde{\sigma} \in (\sigma - \delta_0, \sigma]$,

$$k^{(1)}(\tilde{\sigma}) = \int_0^1 h(t) t^{\tilde{\sigma}-1} dt \in \mathbf{R}_+,$$

then the constant factors in the reverses of (31) and (32) are the best possible.

If $h(t) = 0$ ($t > 1$) in Corollary 1, then

$$h(v_1(x)v_2(y)) = 0 \quad (v_1(x) > \frac{1}{v_2(y)} > 0),$$

and therefore we obtain the following general results:

Corollary 5. Suppose that $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $h(t) \geq 0$, $\sigma \in \mathbf{R}$,

$$k^{(1)}(\sigma) = \int_0^1 h(t) t^{\sigma-1} dt \in \mathbf{R}_+,$$

$0 \leq a_i < b_i \leq \infty$, $v_i'(s) > 0$ ($s \in (a_i, b_i)$), $v_i(a_i^+) = 0$, $v_i(b_i^-) = \infty$ ($i = 1, 2$). If $f(x), g(y) \geq 0$, such that

$$0 < \int_{a_1}^{b_1} \frac{(v_1(x))^{p(1-\sigma)-1}}{(v_1'(x))^{p-1}} f^p(x) dx < \infty$$

and

$$0 < \int_{a_2}^{b_2} \frac{(v_2(y))^{q(1-\sigma)-1}}{(v_2'(y))^{q-1}} g^q(y) dy < \infty,$$

then we have the following equivalent inequalities:

$$\begin{aligned}
& \int_{a_2}^{b_2} \left(\int_{a_1}^{v_1^{-1}(\frac{1}{v_2(y)})} h(v_1(x)v_2(y))f(x)dx \right) g(y)dy \\
&= \int_{a_1}^{b_1} \left(\int_{a_2}^{v_2^{-1}(\frac{1}{v_1(x)})} h(v_1(x)v_2(y))g(y)dy \right) f(x)dx \\
&< k^{(1)}(\sigma) \left[\int_{a_1}^{b_1} \frac{(v_1(x))^{p(1-\sigma)-1}}{(v_1'(x))^{p-1}} f^p(x)dx \right]^{\frac{1}{p}} \\
&\quad \times \left[\int_{a_2}^{b_2} \frac{(v_2(y))^{q(1-\sigma)-1}}{(v_2'(y))^{q-1}} g^q(y)dy \right]^{\frac{1}{q}}, \tag{33}
\end{aligned}$$

$$\begin{aligned}
& \int_{a_2}^{b_2} \frac{v_2'(y)}{(v_2(y))^{1-p\sigma}} \left(\int_{a_1}^{v_1^{-1}(\frac{1}{v_2(y)})} h(v_1(x)v_2(y))f(x)dx \right)^p dy \\
&< (k^{(1)}(\sigma))^p \int_{a_1}^{b_1} \frac{(v_1(x))^{p(1-\sigma)-1}}{(v_1'(x))^{p-1}} f^p(x)dx, \tag{34}
\end{aligned}$$

where, the constant factors $k^{(1)}(\sigma)$ and $(k^{(1)}(\sigma))^p$ are the best possible.

Replacing $p > 1$ by $0 < p < 1$ in the above inequalities, we obtain the equivalent reverses of (33) and (34). If there exists a constant $\delta_0 > 0$, such that for any $\tilde{\sigma} \in (\sigma - \delta_0, \sigma]$,

$$k^{(1)}(\tilde{\sigma}) = \int_0^1 h(t)t^{\tilde{\sigma}-1} dt \in \mathbf{R}_+,$$

then the constant factors in the reverses of (33) and (34) are the best possible.

If $h(t) = 0$ ($0 < t < 1$), then $h(xy) = 0$ ($0 < x < \frac{1}{y}$), and

$$k(\sigma) = \int_0^\infty h(t)t^{\sigma-1} dt = \int_1^\infty h(t)t^{\sigma-1} dt.$$

Setting

$$k^{(2)}(\sigma) := \int_1^\infty h(t)t^{\sigma-1} dt, \tag{35}$$

by Theorem 1 and Theorem 2, we have the following second kind Hardy-type integral inequalities with the non-homogeneous kernel:

Corollary 6. Suppose that $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $h(t) \geq 0$, $\sigma \in \mathbf{R}$,

$$k^{(2)}(\sigma) = \int_1^\infty h(t)t^{\sigma-1} dt \in \mathbf{R}_+.$$

If $f(x), g(y) \geq 0$,

$$0 < \int_0^\infty x^{p(1-\sigma)-1} f^p(x)dx < \infty$$

and

$$0 < \int_0^{\infty} y^{q(1-\sigma)-1} g^q(y) dy < \infty,$$

then we have the following equivalent inequalities:

$$\begin{aligned} \int_0^{\infty} \left(\int_{\frac{1}{y}}^{\infty} h(xy) f(x) dx \right) g(y) dy &= \int_0^{\infty} \left(\int_{\frac{1}{x}}^{\infty} h(xy) g(y) dy \right) f(x) dx \\ &< k^{(2)}(\sigma) \left[\int_0^{\infty} x^{p(1-\sigma)-1} f^p(x) dx \right]^{\frac{1}{p}} \left[\int_0^{\infty} y^{q(1-\sigma)-1} g^q(y) dy \right]^{\frac{1}{q}}, \end{aligned} \quad (36)$$

$$\int_0^{\infty} y^{p\sigma-1} \left(\int_{\frac{1}{y}}^{\infty} h(xy) f(x) dx \right)^p dy < (k^{(2)}(\sigma))^p \int_0^{\infty} x^{p(1-\sigma)-1} f^p(x) dx, \quad (37)$$

where, the constant factors $k^{(2)}(\sigma)$ and $(k^{(2)}(\sigma))^p$ are the best possible.

Replacing $p > 1$ by $0 < p < 1$ in the above inequalities, we derive the equivalent reverses of (36) and (37).

If there exists a constant $\delta_0 > 0$, such that for any $\tilde{\sigma} \in (\sigma - \delta_0, \sigma]$,

$$k^{(2)}(\tilde{\sigma}) = \int_1^{\infty} h(t) t^{\tilde{\sigma}-1} dt \in \mathbf{R}_+,$$

then the constant factors in the reverses of (36) and (37) are the best possible.

If $h(t) = 0$ ($0 < t < 1$) in Corollary 1, then

$$h(v_1(x)v_2(y)) = 0 \quad (0 < v_1(x) < \frac{1}{v_2(y)}).$$

We have the following general results:

Corollary 7. Suppose that $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $h(t) \geq 0$, $\sigma \in \mathbf{R}$,

$$k^{(2)}(\sigma) = \int_1^{\infty} h(t) t^{\sigma-1} dt \in \mathbf{R}_+,$$

$0 \leq a_i < b_i \leq \infty$, $v_i'(s) > 0$ ($s \in (a_i, b_i)$), $v_i(a_i^+) = 0$, $v_i(b_i^-) = \infty$ ($i = 1, 2$). If $f(x), g(y) \geq 0$, such that

$$0 < \int_{a_1}^{b_1} \frac{(v_1(x))^{p(1-\sigma)-1}}{(v_1'(x))^{p-1}} f^p(x) dx < \infty$$

and

$$0 < \int_{a_2}^{b_2} \frac{(v_2(y))^{q(1-\sigma)-1}}{(v_2'(y))^{q-1}} g^q(y) dy < \infty,$$

then we have the following equivalent inequalities:

$$\begin{aligned}
& \int_{a_2}^{b_2} \left(\int_{v_1^{-1}(\frac{1}{v_2(y)})}^{b_1} h(v_1(x)v_2(y))f(x)dx \right) g(y)dy \\
&= \int_{a_1}^{b_1} \left(\int_{v_2^{-1}(\frac{1}{v_1(x)})}^{b_2} h(v_1(x)v_2(y))g(y)dy \right) f(x)dx \\
&< k^{(2)}(\sigma) \left[\int_{a_1}^{b_1} \frac{(v_1(x))^{p(1-\sigma)-1}}{(v_1'(x))^{p-1}} f^p(x)dx \right]^{\frac{1}{p}} \\
&\quad \times \left[\int_{a_2}^{b_2} \frac{(v_2(y))^{q(1-\sigma)-1}}{(v_2'(y))^{q-1}} g^q(y)dy \right]^{\frac{1}{q}}, \tag{38}
\end{aligned}$$

$$\begin{aligned}
& \int_{a_2}^{b_2} \frac{v_2'(y)}{(v_2(y))^{1-p\sigma}} \left(\int_{v_1^{-1}(\frac{1}{v_2(y)})}^{b_1} h(v_1(x)v_2(y))f(x)dx \right)^p dy \\
&< (k^{(2)}(\sigma))^p \int_{a_1}^{b_1} \frac{(v_1(x))^{p(1-\sigma)-1}}{(v_1'(x))^{p-1}} f^p(x)dx, \tag{39}
\end{aligned}$$

where the constant factors $k^{(2)}(\sigma)$ and $(k^{(2)}(\sigma))^p$ are the best possible.

Replacing $p > 1$ by $0 < p < 1$ in the above inequalities, we have the equivalent reverses of (38) and (39). If there exists a constant $\delta_0 > 0$, such that for any $\tilde{\sigma} \in (\sigma - \delta_0, \sigma]$,

$$k^{(2)}(\tilde{\sigma}) = \int_1^{\infty} h(t)t^{\tilde{\sigma}-1}dt \in \mathbf{R}_+,$$

then the constant factors in the reverses of (38) and (39) are the best possible.

Similarly, if $k_\lambda(1, t) = 0$ ($t > 1$), then

$$k_\lambda(x, y) = x^{-\lambda} k_\lambda\left(1, \frac{y}{x}\right) = 0 \quad (y > x > 0),$$

by Corollary 3, we have the following First kind of Hardy-type integral inequalities with the homogeneous kernel:

Corollary 8. Suppose that $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $\mu, \sigma \in \mathbf{R}$, $\mu + \sigma = \lambda$, $k_\lambda(x, y)$ is a homogeneous function of degree $-\lambda$ in \mathbf{R}_+^2 ,

$$k_\lambda^{(1)}(\sigma) = \int_0^1 k_\lambda(1, t)t^{\sigma-1}dt \in \mathbf{R}_+.$$

If $f(x), g(y) \geq 0$,

$$0 < \int_0^\infty x^{p(1-\mu)-1} f^p(x)dx < \infty$$

and

$$0 < \int_0^\infty y^{q(1-\sigma)-1} g^q(y)dy < \infty,$$

then we have the following equivalent inequalities:

$$\begin{aligned} & \int_0^\infty \left(\int_y^\infty k_\lambda(x,y)f(x)dx \right) g(y)dy = \int_0^\infty \left(\int_x^\infty k_\lambda(x,y)g(y)dy \right) f(x)dx \\ & < k_\lambda^{(1)}(\sigma) \left[\int_0^\infty x^{p(1-\mu)-1} f^p(x)dx \right]^{\frac{1}{p}} \left[\int_0^\infty y^{q(1-\sigma)-1} g^q(y)dy \right]^{\frac{1}{q}}, \end{aligned} \quad (40)$$

$$\int_0^\infty y^{p\sigma-1} \left(\int_y^\infty k_\lambda(x,y)f(x)dx \right)^p dy < (k_\lambda^{(1)}(\sigma))^p \int_0^\infty x^{p(1-\mu)-1} f^p(x)dx, \quad (41)$$

where, the constant factors $k_\lambda^{(1)}(\sigma)$ and $(k_\lambda^{(1)}(\sigma))^p$ are the best possible.

Replacing $p > 1$ by $0 < p < 1$ in the above cases, we obtain the equivalent reverses of (28) and (29). If there exists a constant $\delta_0 > 0$, such that for any $\tilde{\sigma} \in (\sigma - \delta_0, \sigma]$,

$$k_\lambda^{(1)}(\tilde{\sigma}) = \int_0^1 k_\lambda(1,t)t^{\tilde{\sigma}-1} dt \in \mathbf{R}_+,$$

then the constant factors in the reverses of (40) and (41) are the best possible.

If $k_\lambda(1,t) = 0$ ($t > 1$) in Corollary 2, then

$$k_\lambda(v_1(x), v_2(y)) = 0 \quad (0 < v_1(x) < v_2(y)),$$

and we have the following general results:

Corollary 9. Suppose that $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $\mu, \sigma \in \mathbf{R}$, $\mu + \sigma = \lambda$, $k_\lambda(x,y)$ is a homogeneous function of degree $-\lambda$ in \mathbf{R}_+^2 ,

$$k_\lambda^{(1)}(\sigma) = \int_0^1 k_\lambda(1,t)t^{\sigma-1} dt \in \mathbf{R}_+,$$

$0 \leq a_i < b_i \leq \infty$, $v_i'(s) > 0$ ($s \in (a_i, b_i)$), $v_i(a_i^+) = 0$, $v_i(b_i^-) = \infty$ ($i = 1, 2$). If $f(x), g(y) \geq 0$, such that

$$0 < \int_{a_1}^{b_1} \frac{(v_1(x))^{p(1-\mu)-1}}{(v_1'(x))^{p-1}} f^p(x)dx < \infty$$

and

$$0 < \int_{a_2}^{b_2} \frac{(v_2(y))^{q(1-\sigma)-1}}{(v_2'(y))^{q-1}} g^q(y)dy < \infty,$$

then we have the following equivalent inequalities:

$$\begin{aligned} & \int_{a_2}^{b_2} \left(\int_{v_1^{-1}(v_2(y))}^{b_1} k_\lambda(v_1(x), v_2(y))f(x)dx \right) g(y)dy \\ & = \int_{a_1}^{b_1} \left(\int_{v_2^{-1}(v_1(x))}^{b_2} k_\lambda(v_1(x), v_2(y))g(y)dy \right) f(x)dx \end{aligned}$$

$$\begin{aligned} &< k_{\lambda}^{(1)}(\sigma) \left[\int_{a_1}^{b_1} \frac{(v_1(x))^{p(1-\mu)-1}}{(v_1'(x))^{p-1}} f^p(x) dx \right]^{\frac{1}{p}} \\ &\quad \times \left[\int_{a_2}^{b_2} \frac{(v_2(y))^{q(1-\sigma)-1}}{(v_2'(y))^{q-1}} g^q(y) dy \right]^{\frac{1}{q}}, \end{aligned} \quad (42)$$

$$\begin{aligned} &\int_{a_2}^{b_2} \frac{v_2'(y)}{(v_2(y))^{1-p\sigma}} \left(\int_{v_1^{-1}(v_2(y))}^{b_1} k_{\lambda}(v_1(x), v_2(y)) f(x) dx \right)^p dy \\ &< (k_{\lambda}^{(1)}(\sigma))^p \int_{a_1}^{b_1} \frac{(v_1(x))^{p(1-\mu)-1}}{(v_1'(x))^{p-1}} f^p(x) dx, \end{aligned} \quad (43)$$

where, the constant factors $k_{\lambda}^{(1)}(\sigma)$ and $(k_{\lambda}^{(1)}(\sigma))^p$ are the best possible.

Replacing $p > 1$ by $0 < p < 1$ in the above cases, we have the equivalent reverses of (42) and (43). If there exists a constant $\delta_0 > 0$, such that for any $\tilde{\sigma} \in (\sigma - \delta_0, \sigma]$,

$$k_{\lambda}^{(1)}(\tilde{\sigma}) = \int_0^1 k_{\lambda}(1, t) t^{\tilde{\sigma}-1} dt \in \mathbf{R}_+,$$

then the constant factors in the reverses of (42) and (43) are the best possible.

Similarly, if $k_{\lambda}(1, t) = 0$ ($0 < t < 1$) in Corollary 3, then

$$k_{\lambda}(x, y) = 0 \quad (x > y > 0),$$

and we have the following second kind Hardy-type integral inequalities with the homogeneous kernel:

Corollary 10. Suppose that $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $\mu, \sigma \in \mathbf{R}$, $\mu + \sigma = \lambda$, $k_{\lambda}(x, y)$ is a homogeneous function of degree $-\lambda$ in \mathbf{R}_+^2 ,

$$k_{\lambda}^{(2)}(\sigma) = \int_1^{\infty} k_{\lambda}(1, t) t^{\sigma-1} dt \in \mathbf{R}_+.$$

If $f(x), g(y) \geq 0$,

$$0 < \int_0^{\infty} x^{p(1-\mu)-1} f^p(x) dx < \infty$$

and

$$0 < \int_0^{\infty} y^{q(1-\sigma)-1} g^q(y) dy < \infty,$$

then we have the following equivalent inequalities:

$$\int_0^{\infty} \left(\int_0^y k_{\lambda}(x, y) f(x) dx \right) g(y) dy = \int_0^{\infty} \left(\int_0^x k_{\lambda}(x, y) g(y) dy \right) f(x) dx$$

$$\langle k_\lambda^{(2)}(\sigma) \left[\int_0^\infty x^{p(1-\mu)-1} f^p(x) dx \right]^{\frac{1}{p}} \left[\int_0^\infty y^{q(1-\sigma)-1} g^q(y) dy \right]^{\frac{1}{q}}, \quad (44)$$

$$\int_0^\infty y^{p\sigma-1} \left(\int_0^y k_\lambda(x,y) f(x) dx \right)^p dy < (k_\lambda^{(2)}(\sigma))^p \int_0^\infty x^{p(1-\mu)-1} f^p(x) dx; \quad (45)$$

where, the constant factors $k_\lambda^{(2)}(\sigma)$ and $(k_\lambda^{(2)}(\sigma))^p$ are the best possible.

Replacing $p > 1$ by $0 < p < 1$ in the above cases, we have the equivalent reverses of (44) and (45). If there exists a constant $\delta_0 > 0$, such that for any $\tilde{\sigma} \in (\sigma - \delta_0, \sigma]$,

$$k_\lambda^{(2)}(\tilde{\sigma}) = \int_1^\infty k_\lambda(1,t) t^{\tilde{\sigma}-1} dt \in \mathbf{R}_+,$$

then the constant factors in the reverses of (44) and (45) are the best possible.

If $k_\lambda(1,t) = 0$ ($0 < t < 1$) in Corollary 2, then

$$k_\lambda(v_1(x), v_2(y)) = 0 \quad (v_1(x) > v_2(y) > 0),$$

we have the following general results:

Corollary 11. Suppose that $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $\mu, \sigma \in \mathbf{R}$, $\mu + \sigma = \lambda$, $k_\lambda(x,y)$ is a homogeneous function of degree $-\lambda$ in \mathbf{R}_+^2 ,

$$k_\lambda^{(2)}(\sigma) = \int_1^\infty k_\lambda(1,t) t^{\sigma-1} dt \in \mathbf{R}_+,$$

$0 \leq a_i < b_i \leq \infty$, $v_i'(s) > 0$ ($s \in (a_i, b_i)$), $v_i(a_i^+) = 0$, $v_i(b_i^-) = \infty$ ($i = 1, 2$). If $f(x), g(y) \geq 0$, such that

$$0 < \int_{a_1}^{b_1} \frac{(v_1(x))^{p(1-\mu)-1}}{(v_1'(x))^{p-1}} f^p(x) dx < \infty$$

and

$$0 < \int_{a_2}^{b_2} \frac{(v_2(y))^{q(1-\sigma)-1}}{(v_2'(y))^{q-1}} g^q(y) dy < \infty,$$

then we have the following equivalent inequalities:

$$\begin{aligned} & \int_{a_2}^{b_2} \left(\int_{a_1}^{v_1^{-1}(v_2(y))} k_\lambda(v_1(x), v_2(y)) f(x) dx \right) g(y) dy \\ &= \int_{a_1}^{b_1} \left(\int_{a_2}^{v_2^{-1}(v_1(x))} k_\lambda(v_1(x), v_2(y)) g(y) dy \right) f(x) dx \\ &< k_\lambda^{(2)}(\sigma) \left[\int_{a_1}^{b_1} \frac{(v_1(x))^{p(1-\mu)-1}}{(v_1'(x))^{p-1}} f^p(x) dx \right]^{\frac{1}{p}} \end{aligned}$$

$$\times \left[\int_{a_2}^{b_2} \frac{(v_2(y))^{q(1-\sigma)-1}}{(v_2'(y))^{q-1}} g^q(y) dy \right]^{\frac{1}{q}}, \quad (46)$$

$$\int_{a_2}^{b_2} \frac{v_2'(y)}{(v_2(y))^{1-p\sigma}} \left(\int_{a_1}^{v_1^{-1}(v_2(y))} k_\lambda(v_1(x), v_2(y)) f(x) dx \right)^p dy \\ < (k_\lambda^{(2)}(\sigma))^p \int_{a_1}^{b_1} \frac{(v_1(x))^{p(1-\mu)-1}}{(v_1'(x))^{p-1}} f^p(x) dx, \quad (47)$$

where, the constant factors $k_\lambda^{(2)}(\sigma)$ and $(k_\lambda^{(2)}(\sigma))^p$ are the best possible.

Replacing $p > 1$ by $0 < p < 1$ in the above inequalities, we have the equivalent reverses of (46) and (47). If there exists a constant $\delta_0 > 0$, such that for any $\tilde{\sigma} \in (\sigma - \delta_0, \sigma]$,

$$k_\lambda^{(2)}(\tilde{\sigma}) = \int_1^\infty k_\lambda(1, t) t^{\tilde{\sigma}-1} dt \in \mathbf{R}_+,$$

then the constant factors in the reverses of (46) and (47) are the best possible.

2.5 Yang-Hilbert-Type Operators and Hardy-Type Operators

Suppose that $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $\mu, \sigma \in \mathbf{R}$, $\mu + \sigma = \lambda$. We set the following functions:

$$\varphi(x) := x^{p(1-\sigma)-1}, \quad \psi(y) := y^{q(1-\sigma)-1}, \quad \phi(x) := x^{p(1-\mu)-1} \quad (x, y \in \mathbf{R}_+),$$

from which we obtain that $\psi^{1-p}(y) = y^{p\sigma-1}$.

Define the following real normed linear space:

$$L_{p,\varphi}(\mathbf{R}_+) := \left\{ f : \|f\|_{p,\varphi} := \left\{ \int_0^\infty \varphi(x) |f(x)|^p dx \right\}^{\frac{1}{p}} < \infty \right\}.$$

Therefore,

$$L_{p,\psi^{1-p}}(\mathbf{R}_+) = \left\{ h : \|h\|_{p,\psi^{1-p}} := \left\{ \int_0^\infty \psi^{1-p}(y) |h(y)|^p dy \right\}^{\frac{1}{p}} < \infty \right\},$$

$$L_{p,\phi}(\mathbf{R}_+) = \left\{ g : \|g\|_{p,\phi} := \left\{ \int_0^\infty \phi(x) |g(x)|^p dx \right\}^{\frac{1}{p}} < \infty \right\}.$$

(a) In view of Theorem 1, for $f \in L_{p,\varphi}(\mathbf{R}_+)$, setting

$$H_1(y) := \int_0^\infty h(xy) |f(x)| dx \quad (y \in \mathbf{R}_+),$$

by (19), we have

$$\|H_1\|_{p,\psi^{1-p}} := \left(\int_0^\infty \psi^{1-p}(y) H_1^p(y) dy \right)^{\frac{1}{p}} < k(\sigma) \|f\|_{p,\varphi} < \infty. \quad (48)$$

Definition 3. Let us define the Yang-Hilbert-type integral operator with the non-homogeneous kernel $T_1 : L_{p,\varphi}(\mathbf{R}_+) \rightarrow L_{p,\psi^{1-p}}(\mathbf{R}_+)$ as follows:

For any $f \in L_{p,\varphi}(\mathbf{R}_+)$, there exists a unique representation $T_1 f = H_1 \in L_{p,\psi^{1-p}}(\mathbf{R}_+)$, satisfying

$$T_1 f(y) = H_1(y),$$

for any $y \in \mathbf{R}_+$.

In view of (48), it follows that

$$\|T_1 f\|_{p,\psi^{1-p}} = \|H_1\|_{p,\psi^{1-p}} \leq k(\sigma) \|f\|_{p,\varphi}$$

and then the operator T_1 is bounded satisfying

$$\|T_1\| = \sup_{f \neq \theta \in L_{p,\varphi}(\mathbf{R}_+)} \frac{\|T_1 f\|_{p,\psi^{1-p}}}{\|f\|_{p,\varphi}} \leq k(\sigma).$$

Since the constant factor $k(\sigma)$ in (48) is the best possible, we have

$$\|T_1\| = k(\sigma) = \int_0^\infty h(t) t^{\sigma-1} dt. \quad (49)$$

If we define the formal inner product of $T_1 f$ and g as

$$(T_1 f, g) := \int_0^\infty \left(\int_0^\infty h(xy) f(x) dx \right) g(y) dy = \int_0^\infty \int_0^\infty h(xy) f(x) g(y) dx dy,$$

then we can rewrite (18) and (19) as follows:

$$(T_1 f, g) < \|T_1\| \cdot \|f\|_{p,\varphi} \|g\|_{q,\psi}, \quad \|T_1 f\|_{p,\psi^{1-p}} < \|T_1\| \cdot \|f\|_{p,\varphi}.$$

(b) In view of Corollary 4, for $f \in L_{p,\varphi}(\mathbf{R}_+)$, setting

$$H_1^{(1)}(y) := \int_0^{\frac{1}{y}} h(xy) |f(x)| dx \quad (y \in \mathbf{R}_+),$$

by (32), we obtain

$$\|H_1^{(1)}\|_{p,\psi^{1-p}} := \left(\int_0^\infty \psi^{1-p}(y) (H_1^{(1)}(y))^p dy \right)^{\frac{1}{p}} < k^{(1)}(\sigma) \|f\|_{p,\varphi} < \infty. \quad (50)$$

Definition 4. Define the Hardy-type integral operator of the first kind with the non-homogeneous kernel

$$T_1^{(1)} : L_{p,\varphi}(\mathbf{R}_+) \rightarrow L_{p,\psi^{1-p}}(\mathbf{R}_+)$$

as follows:

For any $f \in L_{p,\varphi}(\mathbf{R}_+)$, there exists a unique representation $T_1^{(1)}f = H_1^{(1)} \in L_{p,\psi^{1-p}}(\mathbf{R}_+)$, satisfying

$$T_1^{(1)}f(y) = H_1^{(1)}(y),$$

for any $y \in \mathbf{R}_+$.

In view of (50), it follows that

$$\|T_1^{(1)}f\|_{p,\psi^{1-p}} = \|H_1^{(1)}\|_{p,\psi^{1-p}} \leq k^{(1)}(\sigma)\|f\|_{p,\varphi}$$

and thus the operator $T_1^{(1)}$ is bounded satisfying

$$\|T_1^{(1)}\| = \sup_{f(\neq\theta) \in L_{p,\varphi}(\mathbf{R}_+)} \frac{\|T_1^{(1)}f\|_{p,\psi^{1-p}}}{\|f\|_{p,\varphi}} \leq k^{(1)}(\sigma).$$

Since the constant factor $k^{(1)}(\sigma)$ in (50) is the best possible, we have

$$\|T_1^{(1)}\| = k^{(1)}(\sigma) = \int_0^1 h(t)t^{\sigma-1} dt. \tag{51}$$

Setting the formal inner product of $T_1^{(1)}f$ and g as

$$(T_1^{(1)}f, g) = \int_0^\infty \left(\int_0^{\frac{1}{y}} h(xy)f(x)dx \right) g(y)dy,$$

we can rewrite (31) and (32) as follows:

$$(T_1^{(1)}f, g) < \|T_1^{(1)}\| \cdot \|f\|_{p,\varphi} \|g\|_{q,\psi}, \quad \|T_1^{(1)}f\|_{p,\psi^{1-p}} < \|T_1^{(1)}\| \cdot \|f\|_{p,\varphi}. \tag{52}$$

(c) In view of Corollary 6, for $f \in L_{p,\varphi}(\mathbf{R}_+)$, setting

$$H_1^{(2)}(y) := \int_{\frac{1}{y}}^\infty h(xy)|f(x)|dx \quad (y \in \mathbf{R}_+),$$

by (37), we have

$$\|H_1^{(2)}\|_{p,\psi^{1-p}} := \left(\int_0^\infty \psi^{1-p}(y)(H_1^{(2)}(y))^p dy \right)^{\frac{1}{p}} < k^{(2)}(\sigma)\|f\|_{p,\varphi} < \infty. \tag{53}$$

Definition 5. Define the second kind Hardy-type integral operator with the non-homogeneous kernel

$$T_1^{(2)} : L_{p,\varphi}(\mathbf{R}_+) \rightarrow L_{p,\psi^{1-p}}(\mathbf{R}_+)$$

as follows:

For any $f \in L_{p,\phi}(\mathbf{R}_+)$, there exists a unique representation $T_1^{(2)}f = H_1^{(2)} \in L_{p,\psi^{1-p}}(\mathbf{R}_+)$, satisfying

$$T_1^{(2)}f(y) = H_1^{(2)}(y),$$

for any $y \in \mathbf{R}_+$.

In view of (37), it follows that

$$\|T_1^{(2)}f\|_{p,\psi^{1-p}} = \|H_1^{(2)}\|_{p,\psi^{1-p}} \leq k^{(2)}(\sigma)\|f\|_{p,\phi}$$

and hence the operator $T_1^{(2)}$ is bounded satisfying

$$\|T_1^{(2)}\| = \sup_{f(\neq\theta) \in L_{p,\phi}(\mathbf{R}_+)} \frac{\|T_1^{(2)}f\|_{p,\psi^{1-p}}}{\|f\|_{p,\phi}} \leq k^{(2)}(\sigma).$$

Since the constant factor $k^{(2)}(\sigma)$ in (53) is the best possible, we have

$$\|T_1^{(2)}\| = k^{(2)}(\sigma) = \int_1^\infty h(t)t^{\sigma-1} dt. \quad (54)$$

Setting the formal inner product of $T_1^{(2)}f$ and g as

$$(T_1^{(2)}f, g) = \int_0^\infty \left(\int_{\frac{1}{y}}^\infty h(xy)f(x)dx \right) g(y)dy,$$

we can rewrite (36) and (37) as follows:

$$(T_1^{(2)}f, g) < \|T_1^{(2)}\| \cdot \|f\|_{p,\phi} \|g\|_{q,\psi}, \quad \|T_1^{(2)}f\|_{p,\psi^{1-p}} < \|T_1^{(2)}\| \cdot \|f\|_{p,\phi}. \quad (55)$$

(d) In view of Corollary 3, for $f \in L_{p,\phi}(\mathbf{R}_+)$, setting

$$H_2(y) := \int_0^\infty k_\lambda(x,y)|f(x)|dx \quad (y \in \mathbf{R}_+),$$

by (29), we have

$$\|H_2\|_{p,\psi^{1-p}} := \left(\int_0^\infty \psi^{1-p}(y)H_2^p(y)dy \right)^{\frac{1}{p}} < k_\lambda(\sigma)\|f\|_{p,\phi} < \infty. \quad (56)$$

Definition 6. Define the Yang-Hilbert-type integral operator with the homogeneous kernel $T_2 : L_{p,\phi}(\mathbf{R}_+) \rightarrow L_{p,\psi^{1-p}}(\mathbf{R}_+)$ as follows:

For any $f \in L_{p,\phi}(\mathbf{R}_+)$, there exists a unique representation $T_2f = H_2 \in L_{p,\psi^{1-p}}(\mathbf{R}_+)$, satisfying

$$T_2f(y) = H_2(y),$$

for any $y \in \mathbf{R}_+$.

In view of (56), it follows that

$$\|T_2 f\|_{p, \psi^{1-p}} = \|H_2\|_{p, \psi^{1-p}} \leq k_\lambda(\sigma) \|f\|_{p, \phi}$$

and thus the operator T_2 is bounded satisfying

$$\|T_2\| = \sup_{f(\neq \theta) \in L_{p, \phi}(\mathbf{R}_+)} \frac{\|T_2 f\|_{p, \psi^{1-p}}}{\|f\|_{p, \phi}} \leq k_\lambda(\sigma).$$

Since the constant factor $k_\lambda(\sigma)$ in (56) is the best possible, we have

$$\|T_2\| = k_\lambda(\sigma) = \int_0^\infty k_\lambda(1, t) t^{\sigma-1} dt. \quad (57)$$

Setting the formal inner product of $T_2 f$ and g as

$$(T_2 f, g) = \int_0^\infty \left(\int_0^\infty k_\lambda(x, y) f(x) dx \right) g(y) dy = \int_0^\infty \int_0^\infty k_\lambda(x, y) f(x) g(y) dx dy,$$

we can rewrite (28) and (29) as follows:

$$(T_2 f, g) < \|T_2\| \cdot \|f\|_{p, \phi} \|g\|_{q, \psi}, \quad \|T_2 f\|_{p, \psi^{1-p}} < \|T_2\| \cdot \|f\|_{p, \phi}.$$

(e) Due to Corollary 8, for $f \in L_{p, \phi}(\mathbf{R}_+)$,

$$H_2^{(1)}(y) := \int_y^\infty k_\lambda(x, y) |f(x)| dx \quad (y \in \mathbf{R}_+),$$

by (41), we have

$$\|H_2^{(1)}\|_{p, \psi^{1-p}} := \left(\int_0^\infty \psi^{1-p}(y) (H_2^{(1)}(y))^p dy \right)^{\frac{1}{p}} < k_\lambda^{(1)}(\sigma) \|f\|_{p, \phi} < \infty. \quad (58)$$

Definition 7. Define the Hardy-type integral operator of the first kind, with the homogeneous kernel $T_2^{(1)} : L_{p, \phi}(\mathbf{R}_+) \rightarrow L_{p, \psi^{1-p}}(\mathbf{R}_+)$ as follows:

For any $f \in L_{p, \phi}(\mathbf{R}_+)$, there exists a unique representation $T_2^{(1)} f = H_2^{(1)} \in L_{p, \psi^{1-p}}(\mathbf{R}_+)$, satisfying

$$T_2^{(1)} f(y) = H_2^{(1)}(y),$$

for any $y \in \mathbf{R}_+$.

By (41), it follows that

$$\|T_2^{(1)} f\|_{p, \psi^{1-p}} = \|H_2^{(1)}\|_{p, \psi^{1-p}} \leq k_\lambda^{(1)}(\sigma) \|f\|_{p, \phi}$$

and then the operator $T_2^{(1)}$ is bounded satisfying

$$\|T_2^{(1)}\| = \sup_{f(\neq\theta) \in L_{p,\phi}(\mathbf{R}_+)} \frac{\|T_2^{(1)}f\|_{p,\psi^{1-p}}}{\|f\|_{p,\phi}} \leq k_\lambda^{(1)}(\sigma).$$

Since the constant factor $k_\lambda^{(1)}(\sigma)$ in (58) is the best possible, we have

$$\|T_2^{(1)}\| = k_\lambda^{(1)}(\sigma) = \int_0^1 k_\lambda(1,t)t^{\sigma-1}dt. \quad (59)$$

Setting the formal inner product of $T_2^{(1)}f$ and g as

$$(T_2^{(1)}f, g) = \int_0^\infty \left(\int_y^\infty k_\lambda(x,y)f(x)dx \right) g(y)dy,$$

we can rewrite (40) and (41) as follows:

$$(T_2^{(1)}f, g) < \|T_2^{(1)}\| \cdot \|f\|_{p,\phi} \|g\|_{q,\psi}, \quad \|T_2^{(1)}f\|_{p,\psi^{1-p}} < \|T_2^{(1)}\| \cdot \|f\|_{p,\phi}.$$

(f) By Corollary 10, for $f \in L_{p,\phi}(\mathbf{R}_+)$,

$$H_2^{(2)}(y) := \int_0^y k_\lambda(x,y)|f(x)|dx \quad (y \in \mathbf{R}_+),$$

and by (45), we have

$$\|H_2^{(2)}\|_{p,\psi^{1-p}} := \left(\int_0^\infty \psi^{1-p}(y)(H_2^{(2)}(y))^p dy \right)^{\frac{1}{p}} < k_\lambda^{(2)}(\sigma) \|f\|_{p,\phi} < \infty. \quad (60)$$

Definition 8. Define the Hardy-type integral operator of the second kind with the homogeneous kernel $T_2^{(2)} : L_{p,\phi}(\mathbf{R}_+) \rightarrow L_{p,\psi^{1-p}}(\mathbf{R}_+)$ as follows:

For any $f \in L_{p,\phi}(\mathbf{R}_+)$, there exists a unique representation $T_2^{(2)}f = H_2^{(2)} \in L_{p,\psi^{1-p}}(\mathbf{R}_+)$, satisfying

$$T_2^{(2)}f(y) = H_2^{(2)}(y),$$

for any $y \in \mathbf{R}_+$.

In view of (45), it follows that

$$\|T_2^{(2)}f\|_{p,\psi^{1-p}} = \|H_2^{(2)}\|_{p,\psi^{1-p}} \leq k_\lambda^{(2)}(\sigma) \|f\|_{p,\phi}$$

and then the operator $T_2^{(2)}$ is bounded satisfying

$$\|T_2^{(2)}\| = \sup_{f(\neq\theta) \in L_{p,\phi}(\mathbf{R}_+)} \frac{\|T_2^{(2)}f\|_{p,\psi^{1-p}}}{\|f\|_{p,\phi}} \leq k_\lambda^{(2)}(\sigma).$$

Since the constant factor $k_\lambda^{(2)}(\sigma)$ in (60) is the best possible, we have

$$\|T_2^{(2)}\| = k_\lambda^{(2)}(\sigma) = \int_1^\infty k_\lambda(1,t)t^{\sigma-1}dt. \tag{61}$$

Setting the formal inner product of $T_2^{(2)}f$ and g as

$$(T_2^{(2)}f, g) = \int_0^\infty \left(\int_0^y k_\lambda(x,y)f(x)dx \right) g(y)dy,$$

we can rewrite (44) and (45) as follows:

$$(T_2^{(2)}f, g) < \|T_2^{(2)}\| \cdot \|f\|_{p,\phi} \|g\|_{q,\psi}, \|T_2^{(2)}f\|_{p,\psi^{1-p}} < \|T_2^{(2)}\| \cdot \|f\|_{p,\phi}.$$

2.6 Some Examples

Example 1. (a) Set

$$h(t) = k_\lambda(1,t) = \frac{1}{(1+t)^\lambda} \quad (\mu, \sigma > 0, \mu + \sigma = \lambda).$$

Then we have the kernels

$$h(xy) = \frac{1}{(1+xy)^\lambda}, k_\lambda(x,y) = \frac{1}{(x+y)^\lambda}$$

and obtain the constant factors

$$k(\sigma) = k_\lambda(\sigma) = \int_0^\infty \frac{t^{\sigma-1}}{(1+t)^\lambda} dt = B(\mu, \sigma) \in \mathbf{R}_+.$$

By (49) and (57), we have $\|T_1\| = \|T_2\| = B(\mu, \sigma)$.

(b) Set

$$h(t) = k_\lambda(1,t) = \frac{-\ln t}{1-t^\lambda} \quad (\mu, \sigma > 0, \mu + \sigma = \lambda).$$

Then we have the kernels

$$h(xy) = \frac{\ln(xy)}{(xy)^\lambda - 1}, k_\lambda(x,y) = \frac{\ln(x/y)}{x^\lambda - y^\lambda}$$

and obtain the constant factors

$$k(\sigma) = k_\lambda(\sigma) = \int_0^\infty \frac{(\ln t)t^{\sigma-1}}{t^\lambda - 1} dt$$

$$= \frac{1}{\lambda^2} \int_0^\infty \frac{(\ln u) u^{(\sigma/\lambda)-1}}{u-1} du = \left[\frac{\pi}{\lambda \sin \pi(\sigma/\lambda)} \right]^2 \in \mathbf{R}_+.$$

In view of (49) and (57), we have

$$\|T_1\| = \|T_2\| = \left[\frac{\pi}{\lambda \sin \pi(\sigma/\lambda)} \right]^2.$$

(c) Set

$$h(t) = k_\lambda(1, t) = \frac{|\ln t|^\beta}{(\max\{1, t\})^\lambda} \quad (\beta > -1, \mu, \sigma > 0, \mu + \sigma = \lambda).$$

Then we have the kernels

$$h(xy) = \frac{|\ln(xy)|^\beta}{(\max\{1, xy\})^\lambda}, \quad k_\lambda(x, y) = \frac{|\ln(x/y)|^\beta}{(\max\{x, y\})^\lambda}$$

and by using the formula (cf. [9])

$$\Gamma(\alpha) := \int_0^\infty e^{-t} t^{\alpha-1} dt \quad (\alpha > 0)$$

we obtain the following constant factors

$$\begin{aligned} k(\sigma) = k_\lambda(\sigma) &= \int_0^\infty \frac{|\ln t|^\beta t^{\sigma-1}}{(\max\{1, t\})^\lambda} dt \\ &= \int_0^1 (-\ln t)^\beta t^{\sigma-1} dt + \int_1^\infty \frac{(\ln t)^\beta t^{\sigma-1}}{t^\lambda} dt \\ &= \int_0^1 (-\ln t)^\beta (t^{\sigma-1} + t^{\mu-1}) dt = \left(\frac{1}{\sigma^{\beta+1}} + \frac{1}{\mu^{\beta+1}} \right) \int_0^\infty v^\beta e^{-v} dv \\ &= \left(\frac{1}{\sigma^{\beta+1}} + \frac{1}{\mu^{\beta+1}} \right) \Gamma(\beta+1) \in \mathbf{R}_+. \end{aligned}$$

By (49) and (57), we have

$$\|T_1\| = \|T_2\| = \left(\frac{1}{\sigma^{\beta+1}} + \frac{1}{\mu^{\beta+1}} \right) \Gamma(\beta+1).$$

Due to (51) and (59), we have

$$\|T_1^{(1)}\| = \|T_2^{(1)}\| = \frac{1}{\sigma^{\beta+1}} \Gamma(\beta+1),$$

and by (54) and (61), it follows that

$$\|T_1^{(2)}\| = \|T_2^{(2)}\| = \frac{1}{\mu^{\beta+1}} \Gamma(\beta+1).$$

(d) Set

$$h(t) = k_\lambda(1, t) = \frac{|\ln t|^\beta}{(\min\{1, t\})^\lambda} \quad (\beta > -1, \mu, \sigma < 0, \mu + \sigma = \lambda).$$

Then we have the kernels

$$h(xy) = \frac{|\ln(xy)|^\beta}{(\min\{1, xy\})^\lambda}, \quad k_\lambda(x, y) = \frac{|\ln(x/y)|^\beta}{(\min\{x, y\})^\lambda}$$

and obtain the constant factors

$$\begin{aligned} k(\sigma) &= k_\lambda(\sigma) = \int_0^\infty \frac{|\ln t|^\beta t^{\sigma-1}}{(\min\{1, t\})^\lambda} dt \\ &= \int_0^1 \frac{(-\ln t)^\beta t^{\sigma-1}}{t^\lambda} dt + \int_1^\infty (\ln t)^\beta t^{\sigma-1} dt \\ &= \int_0^1 (-\ln t)^\beta (t^{-\mu-1} + t^{-\sigma-1}) dt = \left[\frac{1}{(-\mu)^{\beta+1}} \frac{1}{(-\sigma)^{\beta+1}} \right] \int_0^\infty v^\beta e^{-v} dv \\ &= \left[\frac{1}{(-\mu)^{\beta+1}} + \frac{1}{(-\sigma)^{\beta+1}} \right] \Gamma(\beta + 1) \in \mathbf{R}_+. \end{aligned}$$

In view of (49) and (57), we have

$$\|T_1\| = \|T_2\| = \left[\frac{1}{(-\mu)^{\beta+1}} + \frac{1}{(-\sigma)^{\beta+1}} \right] \Gamma(\beta + 1).$$

By (51) and (59), we have

$$\|T_1^{(1)}\| = \|T_2^{(1)}\| = \frac{1}{(-\mu)^{\beta+1}} \Gamma(\beta + 1),$$

and by (54) and (61), it follows that

$$\|T_1^{(2)}\| = \|T_2^{(2)}\| = \frac{1}{(-\sigma)^{\beta+1}} \Gamma(\beta + 1).$$

(e) Set

$$h(t) = k_\lambda(1, t) = \frac{|\ln t|}{1+t^\lambda} \quad (\mu, \sigma > 0, \mu + \sigma = \lambda).$$

Then we have the kernels

$$h(xy) = \frac{|\ln(xy)|}{1+(xy)^\lambda}, \quad k_\lambda(x, y) = \frac{|\ln(x/y)|}{x^\lambda + y^\lambda}$$

and obtain the constant factors

$$\begin{aligned}
k(\sigma) &= k_\lambda(\sigma) = \int_0^\infty \frac{|\ln t| t^{\sigma-1}}{1+t^\lambda} dt \\
&= \int_0^1 \frac{(-\ln t) t^{\sigma-1}}{t^\lambda + 1} dt + \int_1^\infty \frac{(\ln t) t^{\sigma-1}}{t^\lambda + 1} dt = \int_0^1 \frac{(-\ln t)(t^{\sigma-1} + t^{\mu-1})}{t^\lambda + 1} dt \\
&= \int_0^1 (-\ln t) \sum_{k=0}^\infty (-1)^k (t^{k\lambda + \sigma - 1} + t^{k\lambda + \mu - 1}) dt.
\end{aligned}$$

By the fact that

$$\begin{aligned}
&\sum_{k=0}^\infty \int_0^1 |(-1)^k (-\ln t)(t^{k\lambda + \sigma - 1} + t^{k\lambda + \mu - 1})| dt \\
&= \sum_{k=0}^\infty \int_0^1 (-\ln t) \left(\frac{1}{k\lambda + \sigma} dt^{k\lambda + \sigma} + \frac{1}{k\lambda + \mu} dt^{k\lambda + \mu} \right) \\
&= \sum_{k=0}^\infty \left[\frac{1}{(k\lambda + \sigma)^2} + \frac{1}{(k\lambda + \mu)^2} \right] \in \mathbf{R}_+,
\end{aligned}$$

in combination with Theorem 7 (cf. [35], Chapter 5), we obtain

$$\begin{aligned}
k(\sigma) &= k_\lambda(\sigma) = \int_0^1 (-\ln t) \sum_{k=0}^\infty (-1)^k (t^{k\lambda + \sigma - 1} + t^{k\lambda + \mu - 1}) dt \\
&= \sum_{k=0}^\infty \int_0^1 (-1)^k (-\ln t) (t^{k\lambda + \sigma - 1} + t^{k\lambda + \mu - 1}) dt \\
&= \sum_{k=0}^\infty (-1)^k \left[\frac{1}{(k\lambda + \sigma)^2} + \frac{1}{(k\lambda + \mu)^2} \right] \in \mathbf{R}_+.
\end{aligned}$$

By (49) and (57), we have

$$\|T_1\| = \|T_2\| = \sum_{k=0}^\infty (-1)^k \left[\frac{1}{(k\lambda + \sigma)^2} + \frac{1}{(k\lambda + \mu)^2} \right].$$

By (51) and (59), we have

$$\|T_1^{(1)}\| = \|T_2^{(1)}\| = \sum_{k=0}^\infty (-1)^k \frac{1}{(k\lambda + \sigma)^2},$$

and by (54) and (61), it follows that

$$\|T_1^{(2)}\| = \|T_2^{(2)}\| = \sum_{k=0}^\infty (-1)^k \frac{1}{(k\lambda + \mu)^2}.$$

(f) Set

$$h(t) = k_\lambda(t, 1) = \frac{1}{|1-t|^\lambda} \quad (\mu, \sigma > 0, \mu + \sigma = \lambda < 1).$$

Then, we have kernels

$$h(xy) = \frac{1}{|1-xy|^\lambda}, \quad k_\lambda(x,y) = \frac{1}{|x-y|^\lambda}$$

and obtain the constant factors

$$\begin{aligned} k(\sigma) = k_\lambda(\sigma) &= \int_0^\infty \frac{t^{\sigma-1}}{|1-t|^\lambda} dt \\ &= \int_0^1 \frac{t^{\sigma-1} + t^{\mu-1}}{(1-t)^\lambda} dt = B(1-\lambda, \sigma) + B(1-\lambda, \mu) \in \mathbf{R}_+. \end{aligned}$$

In view of (49) and (57), we obtain

$$\|T_1\| = \|T_2\| = B(1-\lambda, \sigma) + B(1-\lambda, \mu).$$

By (51) and (59), we have

$$\|T_1^{(1)}\| = \|T_2^{(1)}\| = B(1-\lambda, \sigma),$$

and by (54) and (61), it follows that

$$\|T_1^{(2)}\| = \|T_2^{(2)}\| = B(1-\lambda, \mu).$$

For (a)-(f), we can obtain the equivalent inequalities with the kernels and the best possible constant factors in Theorems 1-4. Setting $\delta_0 = \frac{|\sigma|}{2} > 0$, we can still obtain the equivalent reverse inequalities with the kernels and the best possible constant factors in Theorems 1-4.

(g) Set

$$h(t) = k_\lambda(1,t) = \frac{(\min\{t, 1\})^\eta}{(\max\{t, 1\})^{\lambda+\eta}} \quad (\eta > -\min\{\mu, \sigma\}, \mu + \sigma = \lambda).$$

Then we have the kernels

$$h(xy) = \frac{(\min\{1, xy\})^\eta}{(\max\{1, xy\})^{\lambda+\eta}}, \quad k_\lambda(x,y) = \frac{(\min\{x, y\})^\eta}{(\max\{x, y\})^{\lambda+\eta}}$$

and obtain the constant factors

$$\begin{aligned} k(\sigma) = k_\lambda(\sigma) &= \int_0^\infty \frac{(\min\{1, t\})^\eta t^{\sigma-1}}{(\max\{1, t\})^{\lambda+\eta}} dt \\ &= \int_0^1 t^\eta t^{\sigma-1} dt + \int_1^\infty \frac{t^{\sigma-1}}{t^{\lambda+\eta}} dt = \frac{1}{\sigma + \eta} + \frac{1}{\mu + \eta} \\ &= \frac{\lambda + 2\eta}{(\sigma + \eta)(\mu + \eta)} \in \mathbf{R}_+. \end{aligned}$$

In view of (49) and (57), we get

$$\|T_1\| = \|T_2\| = \frac{\lambda + 2\eta}{(\sigma + \eta)(\mu + \eta)}.$$

By (51) and (59), we have

$$\|T_1^{(1)}\| = \|T_2^{(1)}\| = \frac{1}{\sigma + \eta},$$

and by (54) and (61), it follows that

$$\|T_1^{(2)}\| = \|T_2^{(2)}\| = \frac{1}{\mu + \eta}.$$

Then we can derive the equivalent inequalities with the kernels and the best possible constant factors in Theorems 1-4. Setting $\delta_0 = \frac{\sigma + \eta}{2} > 0$, we can still obtain the equivalent reverse inequalities with the kernels and the best possible constant factors in Theorems 1-4.

In particular, (i) for $\eta = 0$,

$$h(t) = k_\lambda(1, t) = \frac{1}{(\max\{1, t\})^\lambda} \quad (\mu, \sigma > 0, \mu + \sigma = \lambda),$$

we have

$$h(xy) = \frac{1}{(\max\{1, xy\})^\lambda}, \quad k_\lambda(x, y) = \frac{1}{(\max\{x, y\})^\lambda}$$

and

$$\|T_1\| = \|T_2\| = \frac{\lambda}{\sigma\mu};$$

(ii) for $\eta = -\lambda$,

$$h(t) = k_\lambda(t, 1) = \frac{1}{(\min\{1, t\})^\lambda} \quad (\mu, \sigma < 0, \mu + \sigma = \lambda),$$

we have

$$h(xy) = \frac{1}{(\min\{1, xy\})^\lambda}, \quad k_\lambda(x, y) = \frac{1}{(\min\{x, y\})^\lambda}$$

and

$$\|T_1\| = \|T_2\| = \frac{-\lambda}{\sigma\mu};$$

(iii) for $\lambda = 0$,

$$h(t) = k_0(1, t) = \left(\frac{\min\{1, t\}}{\max\{1, t\}}\right)^\eta \quad (\eta > |\sigma|),$$

we have

$$h(xy) = \left(\frac{\min\{1, xy\}}{\max\{1, xy\}} \right)^\eta, k_\lambda(x, y) = \left(\frac{\min\{x, y\}}{\max\{x, y\}} \right)^\eta$$

and

$$\|T_1\| = \|T_2\| = \frac{2\eta}{\eta^2 - \sigma^2}.$$

Example 2. (a) Set

$$h(t) = k_0(1, t) = \ln\left(1 + \frac{\rho}{t^\eta}\right) \quad (\rho > 0, 0 < \sigma < \eta).$$

Then we have the kernels

$$h(xy) = \ln\left[1 + \frac{\rho}{(xy)^\eta}\right], \quad k_0(x, y) = \ln\left[1 + \rho\left(\frac{x}{y}\right)^\eta\right]$$

and obtain the constant factors

$$\begin{aligned} k(\sigma) &= k_0(\sigma) = \int_0^\infty t^{\sigma-1} \ln\left(1 + \frac{\rho}{t^\eta}\right) dt \\ &= \frac{1}{\sigma} \int_0^\infty \ln\left(1 + \frac{\rho}{t^\eta}\right) dt^\sigma = \frac{1}{\sigma} \left[t^\sigma \ln\left(1 + \frac{\rho}{t^\eta}\right) \Big|_0^\infty - \int_0^\infty t^\sigma d \ln\left(1 + \frac{\rho}{t^\eta}\right) \right] \\ &= \frac{\eta}{\sigma} \int_0^\infty \frac{t^{\sigma-1}}{(t^\eta/\rho) + 1} dt = \frac{\rho^{\sigma/\eta}}{\sigma} \int_0^\infty \frac{u^{(\sigma/\eta)-1}}{u+1} du \\ &= \frac{\rho^{\sigma/\eta} \pi}{\sigma \sin \pi(\sigma/\eta)} \in \mathbf{R}_+. \end{aligned}$$

In view of (49) and (57), we have

$$\|T_1\| = \|T_2\| = \frac{\rho^{\sigma/\eta} \pi}{\sigma \sin \pi(\sigma/\eta)}.$$

(b) Set

$$h(t) = k_0(1, t) = \arctan\left(\frac{\rho}{t^\eta}\right) \quad (\rho > 0, 0 < \sigma < \eta).$$

Then we have the kernels

$$h(xy) = \arctan\left(\frac{\rho}{(xy)^\eta}\right), \quad k_0(x, y) = \arctan \rho \left(\frac{x}{y}\right)^\eta$$

and obtain the constant factors

$$\begin{aligned} k(\sigma) &= k_0(\sigma) = \int_0^\infty t^{\sigma-1} \arctan\left(\frac{\rho}{t^\eta}\right) dt \\ &= \frac{1}{\sigma} \int_0^\infty \arctan\left(\frac{\rho}{t^\eta}\right) dt^\sigma = \frac{1}{\sigma} \left[t^\sigma \arctan\left(\frac{\rho}{t^\eta}\right) \Big|_0^\infty - \int_0^\infty t^\sigma d \arctan\left(\frac{\rho}{t^\eta}\right) \right] \\ &= \frac{\eta}{\sigma \rho} \int_0^\infty \frac{t^{\eta+\sigma-1}}{(t^{2\eta}/\rho^2) + 1} dt = \frac{\rho^{\sigma/\eta}}{2\sigma} \int_0^\infty \frac{u^{[(\eta+\sigma)/(2\eta)]-1}}{u+1} du \end{aligned}$$

$$= \frac{\rho^{\sigma/\eta} \pi}{2\sigma \sin \pi[(\eta + \sigma)/(2\eta)]} = \frac{\rho^{\sigma/\eta} \pi}{2\sigma \cos \pi[\sigma/(2\eta)]} \in \mathbf{R}_+.$$

By (49) and (57), we have

$$\|T_1\| = \|T_2\| = \frac{\rho^{\sigma/\eta} \pi}{2\sigma \cos \pi[\sigma/(2\eta)]}.$$

(c) Set

$$h(t) = k_0(1, t) = e^{-\rho t^\eta} \quad (\rho, \sigma, \eta > 0).$$

Then we have the kernels

$$h(xy) = e^{-\rho(xy)^\eta}, \quad k_0(x, y) = e^{-\rho(\frac{y}{x})^\eta}$$

and obtain the constant factors

$$\begin{aligned} k(\sigma) &= k_0(\sigma) = \int_0^\infty t^{\sigma-1} e^{-\rho t^\eta} dt \\ &= \frac{1}{\eta \rho^{\sigma/\eta}} \int_0^\infty e^{-u} u^{\frac{\sigma}{\eta}-1} du = \frac{1}{\eta \rho^{\sigma/\eta}} \Gamma\left(\frac{\sigma}{\eta}\right) \in \mathbf{R}_+. \end{aligned}$$

In view of (49) and (57), we have

$$\|T_1\| = \|T_2\| = \frac{1}{\eta \rho^{\sigma/\eta}} \Gamma\left(\frac{\sigma}{\eta}\right).$$

Then for (a)-(c), we can obtain the equivalent inequalities with the kernels and the best possible constant factors in Theorems 1-4. Setting $\delta_0 = \frac{\sigma}{\eta} > 0$, we can still obtain the equivalent reverse inequalities with the kernels and the best possible constant factors in Theorems 1-4.

Example 3. (a) Set

$$h(t) = k_0(1, t) = \operatorname{csc} h(\rho t^\eta) = \frac{2}{e^{\rho t^\eta} - e^{-\rho t^\eta}} \quad (\rho > 0, 0 < \eta < \sigma).$$

Where $\operatorname{csc} h(\cdot)$ stands for the hyperbolic cosecant function (cf. [36]). Then we have the kernels

$$h(xy) = \frac{2}{e^{\rho(xy)^\eta} - e^{-\rho(xy)^\eta}}, \quad k_0(x, y) = \frac{2}{e^{\rho(\frac{y}{x})^\eta} - e^{-\rho(\frac{y}{x})^\eta}}.$$

By the Lebesgue term by term integration theorem, we obtain the constant factors

$$\begin{aligned} k(\sigma) &= k_0(\sigma) = \int_0^\infty \frac{2t^{\sigma-1} dt}{e^{\rho t^\eta} - e^{-\rho t^\eta}} = \int_0^\infty \frac{2t^{\sigma-1} dt}{e^{\rho t^\eta} (1 - e^{-2\rho t^\eta})} \\ &= 2 \int_0^\infty t^{\sigma-1} \sum_{k=0}^\infty e^{-(2k+1)\rho t^\eta} dt = 2 \sum_{k=0}^\infty \int_0^\infty t^{\sigma-1} e^{-(2k+1)\rho t^\eta} dt \end{aligned}$$

$$\begin{aligned}
 &= \frac{2}{\eta\rho^{\sigma/\eta}} \sum_{k=0}^{\infty} \frac{1}{(2k+1)^{\sigma/\eta}} \int_0^{\infty} e^{-u} u^{\frac{\sigma}{\eta}-1} du \\
 &= \frac{2}{\eta\rho^{\sigma/\eta}} \Gamma\left(\frac{\sigma}{\eta}\right) \sum_{k=0}^{\infty} \frac{1}{(2k+1)^{\sigma/\eta}} \\
 &= \frac{2}{\eta\rho^{\sigma/\eta}} \Gamma\left(\frac{\sigma}{\eta}\right) \left[\sum_{k=1}^{\infty} \frac{1}{k^{\sigma/\eta}} - \sum_{k=1}^{\infty} \frac{1}{(2k)^{\sigma/\eta}} \right] \\
 &= \frac{2}{\eta\rho^{\sigma/\eta}} \left(1 - \frac{1}{2^{\sigma/\eta}}\right) \Gamma\left(\frac{\sigma}{\eta}\right) \zeta\left(\frac{\sigma}{\eta}\right) \in \mathbf{R}_+,
 \end{aligned}$$

where, $\zeta\left(\frac{\sigma}{\eta}\right) = \sum_{k=1}^{\infty} \frac{1}{k^{\sigma/\eta}}$ ($\zeta(\cdot)$ is Riemann zeta function). In view of (49) and (57), we have

$$\|T_1\| = \|T_2\| = \frac{2}{\eta\rho^{\sigma/\eta}} \left(1 - \frac{1}{2^{\sigma/\eta}}\right) \Gamma\left(\frac{\sigma}{\eta}\right) \zeta\left(\frac{\sigma}{\eta}\right).$$

(b) Set

$$\begin{aligned}
 h(t) &= k_0(1,t) = e^{-\rho t^\eta} \coth(\rho t^\eta) = e^{-\rho t^\eta} \frac{e^{\rho t^\eta} + e^{-\rho t^\eta}}{e^{\rho t^\eta} - e^{-\rho t^\eta}} \\
 &= \frac{1 + e^{-2\rho t^\eta}}{e^{\rho t^\eta} - e^{-\rho t^\eta}} = \frac{e^{-\rho t^\eta} + e^{-3\rho t^\eta}}{1 - e^{-2\rho t^\eta}} \quad (\rho > 0, 0 < \eta < \sigma).
 \end{aligned}$$

Let $\coth(\cdot)$ stand for the hyperbolic cotangent function (cf. [36]). Then we have the kernels

$$h(xy) = \frac{1 + e^{-2\rho(xy)^\eta}}{e^{\rho(xy)^\eta} - e^{-\rho(xy)^\eta}}, \quad k_0(x,y) = \frac{1 + e^{-2\rho\left(\frac{x}{y}\right)^\eta}}{e^{\rho\left(\frac{x}{y}\right)^\eta} - e^{-\rho\left(\frac{x}{y}\right)^\eta}}.$$

By the Lebesgue term by term integration theorem, we obtain the constant factors

$$\begin{aligned}
 k(\sigma) &= k_0(\sigma) = \int_0^{\infty} \frac{(e^{-\rho t^\eta} + e^{-3\rho t^\eta})t^{\sigma-1}}{1 - e^{-2\rho t^\eta}} dt \\
 &= \int_0^{\infty} t^{\sigma-1} \sum_{k=0}^{\infty} (e^{-(2k+1)\rho t^\eta} + e^{-(2k+3)\rho t^\eta}) dt \\
 &= \frac{1}{\eta\rho^{\sigma/\eta}} \sum_{k=0}^{\infty} \left[\frac{1}{(2k+1)^{\sigma/\eta}} + \frac{1}{(2k+3)^{\sigma/\eta}} \right] \int_0^{\infty} e^{-u} u^{\frac{\sigma}{\eta}-1} du \\
 &= \frac{1}{\eta\rho^{\sigma/\eta}} \Gamma\left(\frac{\sigma}{\eta}\right) \left[2 \sum_{k=0}^{\infty} \frac{1}{(2k+1)^{\sigma/\eta}} - 1 \right] \\
 &= \frac{1}{\eta\rho^{\sigma/\eta}} \Gamma\left(\frac{\sigma}{\eta}\right) \left[\left(2 - \frac{1}{2^{(\sigma/\eta)-1}}\right) \zeta\left(\frac{\sigma}{\eta}\right) - 1 \right] \in \mathbf{R}_+.
 \end{aligned}$$

In view of (49) and (57), we have

$$\|T_1\| = \|T_2\| = \frac{1}{\eta\rho^{\sigma/\eta}}\Gamma\left(\frac{\sigma}{\eta}\right)\left[\left(2 - \frac{1}{2^{(\sigma/\eta)-1}}\right)\zeta\left(\frac{\sigma}{\eta}\right) - 1\right].$$

Then for (a)-(b), we can obtain the equivalent inequalities with the kernels and the best possible constant factors in Theorems 1-4. Setting $\delta_0 = \frac{\sigma-\eta}{2} > 0$, we can still obtain the equivalent reverse inequalities with the kernels and the best possible constant factors in Theorems 1-4.

(c) Set

$$h(t) = k_0(1, t) = \operatorname{sech}(\rho t^\eta) = \frac{2}{e^{\rho t^\eta} + e^{-\rho t^\eta}} \quad (\rho, \eta, \sigma > 0).$$

Let $\operatorname{sech}(\cdot)$ stand for the hyperbolic secant function (cf. [36]). Then we have the kernels

$$h(xy) = \frac{2}{e^{\rho(xy)^\eta} + e^{-\rho(xy)^\eta}}, \quad k_0(x, y) = \frac{2}{e^{\rho(\frac{y}{x})^\eta} + e^{-\rho(\frac{y}{x})^\eta}}.$$

By the Lebesgue term by term integration theorem, we obtain the constant factors

$$\begin{aligned} k(\sigma) &= k_0(\sigma) = \int_0^\infty \frac{2t^{\sigma-1} dt}{e^{\rho t^\eta} + e^{-\rho t^\eta}} = \int_0^\infty \frac{2t^{\sigma-1} dt}{e^{\rho t^\eta} (1 + e^{-2\rho t^\eta})} \\ &= 2 \int_0^\infty t^{\sigma-1} \sum_{k=0}^\infty (-1)^k e^{-(2k+1)\rho t^\eta} dt \\ &= 2 \int_0^\infty t^{\sigma-1} \sum_{k=0}^\infty [e^{-(4k+1)\rho t^\eta} - e^{-(4k+3)\rho t^\eta}] dt \\ &= 2 \sum_{k=0}^\infty \int_0^\infty t^{\sigma-1} [e^{-(4k+1)\rho t^\eta} - e^{-(4k+3)\rho t^\eta}] dt \\ &= 2 \sum_{k=0}^\infty (-1)^k \int_0^\infty t^{\sigma-1} e^{-(2k+1)\rho t^\eta} dt \\ &= \frac{2}{\eta\rho^{\sigma/\eta}} \sum_{k=0}^\infty \frac{(-1)^k}{(2k+1)^{\sigma/\eta}} \int_0^\infty e^{-u} u^{\frac{\sigma}{\eta}-1} du \\ &= \frac{2}{\eta\rho^{\sigma/\eta}} \Gamma\left(\frac{\sigma}{\eta}\right) \xi\left(\frac{\sigma}{\eta}\right) \in \mathbf{R}_+, \end{aligned}$$

where

$$\xi\left(\frac{\sigma}{\eta}\right) = \sum_{k=0}^\infty \frac{(-1)^k}{(2k+1)^{\sigma/\eta}}.$$

In view of (49) and (57), we have

$$\|T_1\| = \|T_2\| = \frac{2}{\eta\rho^{\sigma/\eta}} \Gamma\left(\frac{\sigma}{\eta}\right) \xi\left(\frac{\sigma}{\eta}\right).$$

(d) Set

$$\begin{aligned} h(t) &= k_0(1,t) = e^{-\rho t^\eta} \tanh(\rho t^\eta) = e^{-\rho t^\eta} \frac{e^{\rho t^\eta} - e^{-\rho t^\eta}}{e^{\rho t^\eta} + e^{-\rho t^\eta}} \\ &= \frac{1 - e^{-2\rho t^\eta}}{e^{\rho t^\eta} + e^{-\rho t^\eta}} = \frac{e^{-\rho t^\eta} - e^{-3\rho t^\eta}}{1 + e^{-2\rho t^\eta}} (\rho, \eta, \sigma > 0). \end{aligned}$$

Let $\tanh(\cdot)$ stand for the hyperbolic tangent function (cf. [36]). Then we have the kernels

$$h(xy) = \frac{1 - e^{-2\rho(xy)^\eta}}{e^{\rho(xy)^\eta} + e^{-\rho(xy)^\eta}}, \quad k_0(x,y) = \frac{1 - e^{-2\rho(\frac{y}{x})^\eta}}{e^{\rho(\frac{y}{x})^\eta} + e^{-\rho(\frac{y}{x})^\eta}}.$$

By the Lebesgue term by term integration theorem, we obtain the constant factors

$$\begin{aligned} k(\sigma) &= k_0(\sigma) = \int_0^\infty \frac{(e^{-\rho t^\eta} - e^{-3\rho t^\eta})t^{\sigma-1}}{1 + e^{-2\rho t^\eta}} dt \\ &= \int_0^\infty t^{\sigma-1} \sum_{k=0}^\infty (-1)^k e^{-(2k+1)\rho t^\eta} dt - \int_0^\infty t^{\sigma-1} \sum_{k=0}^\infty (-1)^k e^{-(2k+3)\rho t^\eta} dt \\ &= \int_0^\infty t^{\sigma-1} \sum_{k=0}^\infty [e^{-(4k+1)\rho t^\eta} - e^{-(4k+3)\rho t^\eta}] dt \\ &\quad - \int_0^\infty t^{\sigma-1} \sum_{k=0}^\infty [e^{-(4k+3)\rho t^\eta} - e^{-(4k+5)\rho t^\eta}] dt \\ &= \sum_{k=0}^\infty \left\{ \int_0^\infty t^{\sigma-1} [e^{-(4k+1)\rho t^\eta} - e^{-(4k+3)\rho t^\eta}] dt \right. \\ &\quad \left. - \int_0^\infty t^{\sigma-1} [e^{-(4k+3)\rho t^\eta} - e^{-(4k+5)\rho t^\eta}] dt \right\} \\ &= \frac{1}{\eta \rho^{\sigma/\eta}} \sum_{k=0}^\infty (-1)^k \left[\frac{1}{(2k+1)^{\sigma/\eta}} - \frac{1}{(2k+3)^{\sigma/\eta}} \right] \int_0^\infty e^{-u} u^{\frac{\sigma}{\eta}-1} du \\ &= \frac{1}{\eta \rho^{\sigma/\eta}} \Gamma\left(\frac{\sigma}{\eta}\right) \left[2 \sum_{k=0}^\infty \frac{(-1)^k}{(2k+1)^{\sigma/\eta}} - 1 \right] \\ &= \frac{1}{\eta \rho^{\sigma/\eta}} \Gamma\left(\frac{\sigma}{\eta}\right) \left(2\xi\left(\frac{\sigma}{\eta}\right) - 1 \right) \in \mathbf{R}_+. \end{aligned}$$

In view of (49) and (57), we have

$$\|T_1\| = \|T_2\| = \frac{1}{\eta \rho^{\sigma/\eta}} \Gamma\left(\frac{\sigma}{\eta}\right) \left(2\xi\left(\frac{\sigma}{\eta}\right) - 1 \right).$$

Then for (c)-(d), we can obtain the equivalent inequalities with the kernels and the best possible constant factors in Theorems 1-4. Setting $\delta_0 = \frac{\sigma}{2} > 0$, we can still obtain the equivalent reverse inequalities with the kernels and the best possible constant factors in Theorems 1-4.

Lemma 2. *Let \mathbf{C} stand for the set of complex numbers. If $\mathbf{C}_\infty = \mathbf{C} \cup \{\infty\}$,*

$$z_k \in \mathbf{C} \setminus \{z \mid \operatorname{Re} z \geq 0, \operatorname{Im} z = 0\} \quad (k = 1, 2, \dots, n)$$

are different points, the function $f(z)$ is analytic in \mathbf{C}_∞ except for z_i , $i = 1, 2, \dots, n$, and $z = \infty$ is a zero point of $f(z)$ whose order is not less than 1, then for $\alpha \in \mathbf{R}$, we have

$$\int_0^\infty f(x)x^{\alpha-1} dx = \frac{2\pi i}{1 - e^{2\pi\alpha i}} \sum_{k=1}^n \operatorname{Res}[f(z)z^{\alpha-1}, z_k], \quad (62)$$

where, $0 < \operatorname{Im} \ln z = \arg z < 2\pi$. In particular, if z_k , $k = 1, \dots, n$, are all poles of order 1, setting

$$\varphi_k(z) = (z - z_k)f(z) \quad (\varphi_k(z_k) \neq 0),$$

then

$$\int_0^\infty f(x)x^{\alpha-1} dx = \frac{\pi}{\sin \pi\alpha} \sum_{k=1}^n (-z_k)^{\alpha-1} \varphi_k(z_k). \quad (63)$$

Proof. In view of the theorem (cf. [37], p. 118), we obtain (62). We have

$$\begin{aligned} 1 - e^{2\pi\alpha i} &= 1 - \cos 2\pi\alpha - i \sin 2\pi\alpha \\ &= -2i \sin \pi\alpha (\cos \pi\alpha + i \sin \pi\alpha) = -2ie^{i\pi\alpha} \sin \pi\alpha. \end{aligned}$$

In particular, since

$$f(z)z^{\alpha-1} = \frac{1}{z - z_k} \varphi_k(z)z^{\alpha-1},$$

it is obvious that

$$\operatorname{Res}[f(z)z^{\alpha-1}, -a_k] = z_k^{\alpha-1} \varphi_k(z_k) = -e^{i\pi\alpha} (-z_k)^{\alpha-1} \varphi_k(z_k).$$

Then by (62), we obtain (63). \square

Example 4. (a) Set

$$h(t) = k_\lambda(1, t) = \frac{1}{\prod_{k=1}^s (a_k + t^{\lambda/s})}$$

where $s \in \mathbf{N}$, $0 < a_1 < \dots < a_s$, $\mu, \sigma > 0$, $\mu + \sigma = \lambda$. Then we have the kernels

$$h(xy) = \frac{1}{\prod_{k=1}^s [a_k + (xy)^{\lambda/s}]}, \quad k_\lambda(x, y) = \frac{1}{\prod_{k=1}^s (a_k x^{\lambda/s} + y^{\lambda/s}).}$$

For

$$f(z) = \frac{1}{\prod_{k=1}^s (z + a_k)}, \quad z_k = -a_k,$$

by (63), we get

$$\varphi_k(z_k) = (z + a_k) \frac{1}{\prod_{i=1}^s (z + a_i)} \Big|_{z=-a_k} = \prod_{j=1(j \neq k)}^s \frac{1}{a_j - a_k},$$

and obtain the constant factors

$$\begin{aligned} k(\sigma) &= k_\lambda(\sigma) = \int_0^\infty \frac{t^{\sigma-1} dt}{\prod_{k=1}^s (a_k + t^{\lambda/s})} = \frac{s}{\lambda} \int_0^\infty \frac{u^{(s\sigma/\lambda)-1} du}{\prod_{k=1}^s (u + a_k)} \\ &= \frac{\pi s}{\lambda \sin \pi(s\sigma/\lambda)} \sum_{k=1}^s a_k^{s\sigma/\lambda} \prod_{j=1(j \neq k)}^s \frac{1}{a_j - a_k} \in \mathbf{R}_+. \end{aligned}$$

In view of (49) and (57), we have

$$\|T_1\| = \|T_2\| = \frac{\pi s}{\lambda \sin \pi(s\sigma/\lambda)} \sum_{k=1}^s a_k^{s\sigma/\lambda} \prod_{j=1(j \neq k)}^s \frac{1}{a_j - a_k}.$$

In particular, (i) if $s = 1, a_1 = a, h(t) = k_\lambda(1, t) = \frac{1}{a+t^\lambda} (a, \mu, \sigma > 0, \mu + \sigma = \lambda)$, then we have the kernels $h(xy) = \frac{1}{a+(xy)^\lambda}, k_\lambda(x, y) = \frac{1}{ax^\lambda + y^\lambda}$, and

$$\|T_1\| = \|T_2\| = \frac{\pi}{\lambda \sin \pi(\sigma/\lambda)} a^{\frac{\sigma}{\lambda}-1};$$

(ii) if $s = 2, a_1 = a, a_2 = b$,

$$h(t) = k_\lambda(1, t) = \frac{1}{(a+t^{\lambda/2})(b+t^{\lambda/2})} \quad (0 < a < b, \mu, \sigma > 0, \mu + \sigma = \lambda),$$

then we have the kernels

$$h(xy) = \frac{1}{[a+(xy)^{\lambda/2}][b+(xy)^{\lambda/2}]}, \quad k_\lambda(x, y) = \frac{1}{(ax^{\lambda/2} + y^{\lambda/2})(ax^{\lambda/2} + y^{\lambda/2})},$$

and

$$\|T_1\| = \|T_2\| = \frac{2\pi}{\lambda \sin \pi(2\sigma/\lambda)} \frac{1}{b-a} (a^{\frac{2\sigma}{\lambda}-1} - b^{\frac{2\sigma}{\lambda}-1}).$$

(b) Set

$$h(t) = k_\lambda(1, t) = \frac{1}{t^\lambda + 2ct^{\lambda/2} \cos \gamma + c^2}$$

($c > 0, |\gamma| < \frac{\pi}{2}, \mu, \sigma > 0, \mu, \sigma = \lambda$). Then we have the kernels

$$h(xy) = \frac{1}{(xy)^\lambda + 2c(xy)^{\lambda/2} \cos \gamma + c^2},$$

$$k_\lambda(x, y) = \frac{1}{y^\lambda + 2c(xy)^{\lambda/2} \cos \gamma + c^2 x^\lambda}.$$

By (63), we can find

$$\begin{aligned} k(\sigma) &= k_\lambda(\sigma) = \int_0^\infty \frac{t^{\sigma-1}}{t^\lambda + 2ct^{\lambda/2} \cos \gamma + c^2} dt \\ &= \frac{2}{\lambda} \int_0^\infty \frac{u^{(2\sigma/\lambda)-1} du}{u^2 + 2cu \cos \gamma + c^2} = \frac{2}{\lambda} \int_0^\infty \frac{u^{(2\sigma/\lambda)-1} du}{(u + ce^{i\gamma})(u + ce^{-i\gamma})} \end{aligned}$$

$$\begin{aligned}
&= \frac{2\pi}{\lambda \sin \pi(2\sigma/\lambda)} \left[\frac{(ce^{i\gamma})^{(2\sigma/\lambda)-1}}{c(e^{-i\gamma} - e^{i\gamma})} + \frac{(ce^{-i\gamma})^{(2\sigma/\lambda)-1}}{c(e^{i\gamma} - e^{-i\gamma})} \right] \\
&= \frac{2\pi \sin \gamma(1 - 2\sigma/\lambda)}{\lambda \sin \pi(2\sigma/\lambda) \sin \gamma} c^{\frac{2\sigma}{\lambda}-2} \in \mathbf{R}_+.
\end{aligned}$$

In view of (49) and (57), we have

$$\|T_1\| = \|T_2\| = \frac{2\pi \sin \gamma(1 - 2\sigma/\lambda)}{\lambda \sin \pi(2\sigma/\lambda) \sin \gamma} c^{\frac{2\sigma}{\lambda}-2}.$$

Then for (a)-(b), we can obtain the equivalent inequalities with the kernels and the best possible constant factors in Theorems 1-4. Setting $\delta_0 = \frac{\sigma}{2} > 0$, we can obtain the equivalent reverse inequalities with the kernels and the best possible constant factors in Theorems 1-4.

Remark 4. Setting $p = q = 2$, $\mu = \sigma = \frac{\lambda}{2}$ in Theorem 1 and Corollary 3, in view of Remark 3 and the above results, if $f(x), g(y) \geq 0$, with

$$0 < \int_0^\infty x^{1-\lambda} f^2(x) dx < \infty$$

and

$$0 < \int_0^\infty y^{1-\lambda} g^2(y) dy < \infty,$$

then we obtain the following 8 couples of simpler equivalent inequalities with an independent parameter λ and the best possible constant factors:

(a) For $\lambda > 0$,

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x^\lambda + y^\lambda} dx dy < \frac{\pi}{\lambda} \left(\int_0^\infty x^{1-\lambda} f^2(x) dx \int_0^\infty y^{1-\lambda} g^2(y) dy \right)^{\frac{1}{2}}, \quad (64)$$

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{1 + (xy)^\lambda} dx dy < \frac{\pi}{\lambda} \left(\int_0^\infty x^{1-\lambda} f^2(x) dx \int_0^\infty y^{1-\lambda} g^2(y) dy \right)^{\frac{1}{2}}; \quad (65)$$

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x+y)^\lambda} dx dy < B \left(\frac{\lambda}{2}, \frac{\lambda}{2} \right) \left(\int_0^\infty x^{1-\lambda} f^2(x) dx \int_0^\infty y^{1-\lambda} g^2(y) dy \right)^{\frac{1}{2}}, \quad (66)$$

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(1+xy)^\lambda} dx dy < B \left(\frac{\lambda}{2}, \frac{\lambda}{2} \right) \left(\int_0^\infty x^{1-\lambda} f^2(x) dx \int_0^\infty y^{1-\lambda} g^2(y) dy \right)^{\frac{1}{2}}, \quad (67)$$

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(\max\{x, y\})^\lambda} dx dy < \frac{4}{\lambda} \left(\int_0^\infty x^{1-\lambda} f^2(x) dx \int_0^\infty y^{1-\lambda} g^2(y) dy \right)^{\frac{1}{2}}, \quad (68)$$

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(\max\{xy, 1\})^\lambda} dx dy < \frac{4}{\lambda} \left(\int_0^\infty x^{1-\lambda} f^2(x) dx \int_0^\infty y^{1-\lambda} g^2(y) dy \right)^{\frac{1}{2}}; \quad (69)$$

$$\int_0^\infty \int_0^\infty \frac{|\ln(\frac{x}{y})|f(x)g(y)}{(\max\{x,y\})^\lambda} dx dy < \frac{8}{\lambda^2} \left(\int_0^\infty x^{1-\lambda} f^2(x) dx \int_0^\infty y^{1-\lambda} g^2(y) dy \right)^{\frac{1}{2}}, \quad (70)$$

$$\int_0^\infty \int_0^\infty \frac{|\ln(xy)|f(x)g(y)}{(\max\{1,xy\})^\lambda} dx dy < \frac{8}{\lambda^2} \left(\int_0^\infty x^{1-\lambda} f^2(x) dx \int_0^\infty y^{1-\lambda} g^2(y) dy \right)^{\frac{1}{2}}; \quad (71)$$

$$\int_0^\infty \int_0^\infty \frac{\ln(\frac{x}{y})f(x)g(y)}{x^\lambda - y^\lambda} dx dy < \left(\frac{\pi}{\lambda}\right)^2 \left(\int_0^\infty x^{1-\lambda} f^2(x) dx \int_0^\infty y^{1-\lambda} g^2(y) dy \right)^{\frac{1}{2}}, \quad (72)$$

$$\int_0^\infty \int_0^\infty \frac{\ln(xy)f(x)g(y)}{(xy)^\lambda - 1} dx dy < \left(\frac{\pi}{\lambda}\right)^2 \left(\int_0^\infty x^{1-\lambda} f^2(x) dx \int_0^\infty y^{1-\lambda} g^2(y) dy \right)^{\frac{1}{2}}; \quad (73)$$

(b) for $0 < \lambda < 1$,

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{|x-y|^\lambda} dx dy < 2B\left(1-\lambda, \frac{\lambda}{2}\right) \left(\int_0^\infty x^{1-\lambda} f^2(x) dx \int_0^\infty y^{1-\lambda} g^2(y) dy \right)^{\frac{1}{2}}, \quad (74)$$

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{|1-xy|^\lambda} dx dy < 2B\left(1-\lambda, \frac{\lambda}{2}\right) \left(\int_0^\infty x^{1-\lambda} f^2(x) dx \int_0^\infty y^{1-\lambda} g^2(y) dy \right)^{\frac{1}{2}}; \quad (75)$$

(c) for $\lambda < 0$,

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(\min\{x,y\})^\lambda} dx dy < \frac{-4}{\lambda} \left(\int_0^\infty x^{1-\lambda} f^2(x) dx \int_0^\infty y^{1-\lambda} g^2(y) dy \right)^{\frac{1}{2}}, \quad (76)$$

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(\min\{1,xy\})^\lambda} dx dy < \frac{-4}{\lambda} \left(\int_0^\infty x^{1-\lambda} f^2(x) dx \int_0^\infty y^{1-\lambda} g^2(y) dy \right)^{\frac{1}{2}}; \quad (77)$$

$$\int_0^\infty \int_0^\infty \frac{|\ln(\frac{x}{y})|f(x)g(y)}{(\min\{x,y\})^\lambda} dx dy < \frac{8}{\lambda^2} \left(\int_0^\infty x^{1-\lambda} f^2(x) dx \int_0^\infty y^{1-\lambda} g^2(y) dy \right)^{\frac{1}{2}}, \quad (78)$$

$$\int_0^\infty \int_0^\infty \frac{|\ln(xy)|f(x)g(y)}{(\min\{1,xy\})^\lambda} dx dy < \frac{8}{\lambda^2} \left(\int_0^\infty x^{1-\lambda} f^2(x) dx \int_0^\infty y^{1-\lambda} g^2(y) dy \right)^{\frac{1}{2}}. \quad (79)$$

3 Yang-Hilbert-Type Integral Inequalities in the Whole Plane

In this section, we study some Yang-Hilbert-type integral inequalities in the whole plane with parameters and the best constant factors. The equivalent forms, the reverses, the Hardy-type inequalities, the operator expressions and some particular examples are also discussed.

3.1 Weight Functions and a Lemma

Definition 9. If $\delta \in \{-1, 1\}$, $\sigma \in \mathbf{R}$, $H(t)$ is a non-negative measurable function in \mathbf{R} , define the following weight functions:

$$\omega_\delta(\sigma, y) := |y|^\sigma \int_{-\infty}^{\infty} H(x^\delta y) |x|^{\delta\sigma-1} dx (y \in \mathbf{R} \setminus \{0\}), \quad (80)$$

$$\bar{\omega}_\delta(\sigma, x) := |x|^{\delta\sigma} \int_{-\infty}^{\infty} H(x^\delta y) |y|^{\sigma-1} dy (x \in \mathbf{R} \setminus \{0\}). \quad (81)$$

Setting $t = x^\delta y$ in (80), we obtain $x = y^{-\frac{1}{\delta}} t^{\frac{1}{\delta}}$, $dx = \frac{1}{\delta} y^{-\frac{1}{\delta}} t^{\frac{1}{\delta}-1} dt$ and

$$\begin{aligned} \omega_\delta(\sigma, y) &= |y|^\sigma \int_{-\infty}^{\infty} H(t) |y^{-\frac{1}{\delta}} t^{\frac{1}{\delta}}|^{\delta\sigma-1} |y|^{-\frac{1}{\delta}} |t|^{\frac{1}{\delta}-1} dt \\ &= K(\sigma) := \int_{-\infty}^{\infty} H(t) |t|^{\sigma-1} dt. \end{aligned} \quad (82)$$

Setting $t = x^\delta y$ in (81), we find $y = x^{-\delta} t$, $dy = x^{-\delta} dt$ and

$$\bar{\omega}_\delta(\sigma, x) = |x|^{\delta\sigma} \int_{-\infty}^{\infty} H(t) |x^{-\delta} t|^{\sigma-1} |x|^{-\delta} dt = K(\sigma). \quad (83)$$

Remark 5. We can still get

$$\begin{aligned} K(\sigma) &= \int_{-\infty}^0 H(t) (-t)^{\sigma-1} dt + \int_0^{\infty} H(t) t^{\sigma-1} dt \\ &= \int_0^{\infty} H(-u) u^{\sigma-1} du + \int_0^{\infty} H(t) t^{\sigma-1} dt \\ &= \int_0^{\infty} (H(-t) + H(t)) t^{\sigma-1} dt. \end{aligned} \quad (84)$$

If $H(t) = H(-t)$, then

$$K(\sigma) = 2 \int_0^{\infty} H(t) t^{\sigma-1} dt,$$

and thus we obtain again cases of integrals in the first quadrant. In this section, we assume that $H(t) \neq H(-t)$.

Lemma 3. If $p > 0$ ($p \neq 1$), $\frac{1}{p} + \frac{1}{q} = 1$, $\sigma \in \mathbf{R}$, both $H(t)$ and $f(t)$ are non-negative measurable functions in \mathbf{R} , and $K(\sigma)$ is defined by (83), then, (i) for $p > 1$, we have the following inequality:

$$\begin{aligned} J &:= \int_{-\infty}^{\infty} |y|^{p\sigma-1} \left(\int_{-\infty}^{\infty} H(x^\delta y) f(x) dx \right)^p dy \\ &\leq K^p(\sigma) \int_{-\infty}^{\infty} |x|^{p(1-\delta\sigma)-1} f^p(x) dx; \end{aligned} \quad (85)$$

(ii) for $0 < p < 1$, we have the reverse of (85).

Proof. (i) By Hölder's weighted inequality (cf. [33]) and (80), it follows that

$$\begin{aligned}
& \int_{-\infty}^{\infty} H(x^\delta y) f(x) dx \\
&= \int_{-\infty}^{\infty} H(x^\delta y) \left[\frac{|x|^{(1-\delta\sigma)/q}}{|y|^{(1-\sigma)/p}} f(x) \right] \left[\frac{|y|^{(1-\sigma)/p}}{|x|^{(1-\delta\sigma)/q}} \right] dx \\
&\leq \left[\int_{-\infty}^{\infty} H(x^\delta y) \frac{|x|^{(1-\delta\sigma)p/q}}{|y|^{1-\sigma}} f^p(x) dx \right]^{\frac{1}{p}} \\
&\quad \times \left[\int_{-\infty}^{\infty} H(x^\delta y) \frac{|y|^{(1-\sigma)q/p}}{|x|^{1-\delta\sigma}} dx \right]^{\frac{1}{q}} \\
&= \frac{(\omega_\delta(\sigma, y))^{\frac{1}{q}}}{|y|^{\sigma - \frac{1}{p}}} \left[\int_{-\infty}^{\infty} H(x^\delta y) \frac{|x|^{(1-\delta\sigma)(p-1)}}{|y|^{1-\sigma}} f^p(x) dx \right]^{\frac{1}{p}}. \tag{86}
\end{aligned}$$

Then, by (82) and Fubini's theorem (cf. [34]), we get

$$\begin{aligned}
J &\leq K^{p-1}(\sigma) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} H(x^\delta y) \frac{|x|^{(1-\delta\sigma)(p-1)}}{|y|^{1-\sigma}} f^p(x) dx dy \\
&= K^{p-1}(\sigma) \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} H(x^\delta y) \frac{|x|^{(1-\delta\sigma)(p-1)}}{|y|^{1-\sigma}} dy \right] f^p(x) dx \\
&= K^{p-1}(\sigma) \int_{-\infty}^{\infty} \omega_\delta(\sigma, x) |x|^{p(1-\delta\sigma)-1} f^p(x) dx. \tag{87}
\end{aligned}$$

By (83), we obtain (85).

(ii) For $0 < p < 1$, by the reverse of Hölder's weighted inequality (cf. [33]), we can similarly derive the reverses of (86) and (87). Then we obtain the reverse of (85).

This completes the proof of the lemma.

3.2 Equivalent Inequalities with the Best Possible Constant Factors

Theorem 5. Suppose that $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $\sigma \in \mathbf{R}$, $H(t) \geq 0$, and

$$K(\sigma) = \int_{-\infty}^{\infty} H(t) |t|^{\sigma-1} dt \in \mathbf{R}_+.$$

If $f(x), g(y) \geq 0$, such that

$$0 < \int_{-\infty}^{\infty} |x|^{p(1-\delta\sigma)-1} f^p(x) dx < \infty$$

and

$$0 < \int_0^{\infty} y^{q(1-\sigma)-1} g^q(y) dy < \infty,$$

then we obtain the following equivalent inequalities:

$$\begin{aligned} I &:= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} H(x^\delta y) f(x) g(y) dx dy \\ &< K(\sigma) \left[\int_{-\infty}^{\infty} |x|^{p(1-\delta\sigma)-1} f^p(x) dx \right]^{\frac{1}{p}} \left[\int_0^{\infty} y^{q(1-\sigma)-1} g^q(y) dy \right]^{\frac{1}{q}}, \end{aligned} \quad (88)$$

$$\begin{aligned} J &= \int_{-\infty}^{\infty} |y|^{p\sigma-1} \left(\int_{-\infty}^{\infty} H(x^\delta y) f(x) dx \right)^p dy \\ &< K^p(\sigma) \int_{-\infty}^{\infty} |x|^{p(1-\delta\sigma)-1} f^p(x) dx, \end{aligned} \quad (89)$$

where the constant factors $K(\sigma)$ and $K^p(\sigma)$ are the best possible.

In particular, for $\delta = 1$, we have

$$\begin{aligned} I &:= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} H(xy) f(x) g(y) dx dy \\ &< K(\sigma) \left[\int_{-\infty}^{\infty} |x|^{p(1-\sigma)-1} f^p(x) dx \right]^{\frac{1}{p}} \left[\int_0^{\infty} y^{q(1-\sigma)-1} g^q(y) dy \right]^{\frac{1}{q}}, \end{aligned} \quad (90)$$

$$\begin{aligned} J &= \int_{-\infty}^{\infty} |y|^{p\sigma-1} \left(\int_{-\infty}^{\infty} H(xy) f(x) dx \right)^p dy \\ &< K^p(\sigma) \int_{-\infty}^{\infty} |x|^{p(1-\sigma)-1} f^p(x) dx. \end{aligned} \quad (91)$$

Proof. We shall initially prove that (86) preserves the form of strict inequality for any $y \in \mathbf{R} \setminus \{0\}$. There exist two constants A and B , such that they are not both zero, and (cf. [33])

$$A \frac{|x|^{(1-\delta\sigma)p/q}}{|y|^{1-\sigma}} f^p(x) = B \frac{|y|^{(1-\sigma)q/p}}{|x|^{1-\sigma}} \quad \text{a. e. in } \mathbf{R}.$$

If $A = 0$, then $B = 0$, which is impossible. Suppose that $A \neq 0$. Then it follows that

$$|x|^{p(1-\delta\sigma)-1} f^p(x) = |y|^{(1-\sigma)q} \frac{B}{A|x|} \quad \text{a. e. in } \mathbf{R},$$

which contradicts the fact that

$$0 < \int_{-\infty}^{\infty} |x|^{p(1-\delta\sigma)-1} f^p(x) dx < \infty,$$

in virtue of

$$\int_{-\infty}^{\infty} \frac{1}{|x|} dx = \infty.$$

Hence, both (86) and (87) preserve the forms of strict inequality, and thus we obtain (89).

By Hölder's inequality (cf. [33]), we find

$$\begin{aligned} I &= \int_{-\infty}^{\infty} \left(|y|^{\sigma-\frac{1}{p}} \int_{-\infty}^{\infty} H(x^\delta y) f(x) dx \right) (|y|^{\frac{1}{p}-\sigma} g(y)) dy \\ &\leq J^{\frac{1}{p}} \left[\int_{-\infty}^{\infty} |y|^{q(1-\sigma)-1} g^q(y) dy \right]^{\frac{1}{q}}. \end{aligned} \tag{92}$$

Then by (89), we have (88). On the other hand, assuming that (88) is valid, we set

$$g(y) := |y|^{p\sigma-1} \left(\int_{-\infty}^{\infty} H(x^\delta y) f(x) dx \right)^{p-1}, \quad y \in \mathbf{R}.$$

Then we get

$$J = \int_{-\infty}^{\infty} |y|^{q(1-\sigma)-1} g^q(y) dy.$$

By (85), in view of

$$0 < \int_{-\infty}^{\infty} |x|^{p(1-\delta\sigma)-1} f^p(x) dx < \infty,$$

it follows that $J < \infty$.

If $J = 0$, then, (89) is trivially valid; if $J > 0$, then by (88), we have

$$\begin{aligned} 0 &< \int_{-\infty}^{\infty} |y|^{q(1-\sigma)-1} g^q(y) dy = J = I \\ &< K(\sigma) \left[\int_{-\infty}^{\infty} |x|^{p(1-\delta\sigma)-1} f^p(x) dx \right]^{\frac{1}{p}} \left[\int_{-\infty}^{\infty} |y|^{q(1-\sigma)-1} g^q(y) dy \right]^{\frac{1}{q}}, \\ J^{\frac{1}{p}} &= \left[\int_{-\infty}^{\infty} |y|^{q(1-\sigma)-1} g^q(y) dy \right]^{\frac{1}{p}} < K(\sigma) \left[\int_{-\infty}^{\infty} |x|^{p(1-\delta\sigma)-1} f^p(x) dx \right]^{\frac{1}{p}}, \end{aligned}$$

and then (89) follows, which is equivalent to (88).

For any $n \in \mathbf{N}$, we define two sets

$$E_\delta := \{x \in \mathbf{R}; |x|^\delta \geq 1\}, \quad E_\delta^+ := \{x \in \mathbf{R}_+; x^\delta \geq 1\},$$

and the functions $f_n(x), g_n(y)$ as follows:

$$f_n(x) := \begin{cases} 0, & x \in \mathbf{R} \setminus E_\delta \\ |x|^{\delta(\sigma-\frac{1}{np})-1}, & x \in E_\delta \end{cases} \quad g_n(y) := \begin{cases} |y|^{\sigma+\frac{1}{nq}-1}, & y \in [-1, 1] \\ 0, & y \in \mathbf{R} \setminus [-1, 1] \end{cases}$$

Then we find

$$\begin{aligned}
L_n &:= \left[\int_{-\infty}^{\infty} |x|^{p(1-\delta\sigma)-1} f_n^p(x) dx \right]^{\frac{1}{p}} \left[\int_{-\infty}^{\infty} |y|^{q(1-\sigma)-1} g_n^q(y) dy \right]^{\frac{1}{q}} \\
&= \left(\int_{E_\delta} |x|^{-\frac{\delta}{n}-1} dx \right)^{\frac{1}{p}} \left(\int_{-1}^1 |y|^{\frac{1}{n}-1} dy \right)^{\frac{1}{q}} \\
&= \left(2 \int_{E_\delta^+} x^{-\frac{\delta}{n}-1} dx \right)^{\frac{1}{p}} \left(2 \int_0^1 y^{\frac{1}{n}-1} dy \right)^{\frac{1}{q}} = 2n.
\end{aligned}$$

Setting $Y = -y$, we obtain

$$\begin{aligned}
I(x) &:= \int_{-1}^1 H(x^\delta y) |y|^{\sigma+\frac{1}{nq}-1} dy \\
&= \int_{-1}^1 H((-x)^\delta Y) |Y|^{\sigma+\frac{1}{nq}-1} dY = I(-x),
\end{aligned}$$

and then $I(x)$ is an even function. For $x > 0$, we find

$$\begin{aligned}
I(x) &= \int_{-1}^0 H(x^\delta y) (-y)^{\sigma+\frac{1}{nq}-1} dy + \int_0^1 H(x^\delta y) y^{\sigma+\frac{1}{nq}-1} dy \\
&= x^{-\delta\sigma-\frac{\delta}{nq}} \int_0^{x^\delta} (H(-t) + H(t)) t^{\sigma+\frac{1}{nq}-1} dt.
\end{aligned}$$

By the above results and Fubini's theorem, it follows that

$$\begin{aligned}
I_n &:= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} H(x^\delta y) f_n(x) g_n(y) dx dy \\
&= \int_{E_\delta} |x|^{\delta(\sigma-\frac{1}{np})-1} \left(\int_{-1}^1 H(x^\delta y) |y|^{\sigma+\frac{1}{nq}-1} dy \right) dx \\
&= \int_{E_\delta} |x|^{\delta(\sigma-\frac{1}{np})-1} I(x) dx = 2 \int_{E_\delta^+} x^{\delta(\sigma-\frac{1}{np})-1} I(x) dx \\
&= 2 \int_{E_\delta^+} x^{-\frac{\delta}{n}-1} \left(\int_0^1 (H(-t) + H(t)) t^{\sigma+\frac{1}{nq}-1} dt \right) dx \\
&\quad + 2 \int_{E_\delta^+} x^{-\frac{\delta}{n}-1} \left(\int_1^{x^\delta} (H(-t) + H(t)) t^{\sigma+\frac{1}{nq}-1} dt \right) dx \\
&= 2n \left(\int_0^1 (H(-t) + H(t)) t^{\sigma+\frac{1}{nq}-1} dt \right) \\
&\quad + 2 \int_1^{\infty} \left(\int_{\{x>0; x^\delta \geq t\}} x^{-\frac{\delta}{n}-1} dx \right) (H(-t) + H(t)) t^{\sigma+\frac{1}{nq}-1} dt \\
&= 2n \left[\int_0^1 (H(-t) + H(t)) t^{\sigma+\frac{1}{nq}-1} dt + \int_1^{\infty} (H(-t) + H(t)) t^{\sigma-\frac{1}{np}-1} dt \right].
\end{aligned}$$

If there exists a positive number $k \leq K(\sigma)$, such that (88) is still valid when replacing $K(\sigma)$ by k , then in particular, it follows that

$$\frac{1}{2n}I_n < k \frac{1}{2n}L_n,$$

and

$$\int_0^1 (H(-t) + H(t))t^{\sigma + \frac{1}{nq} - 1} dt + \int_1^\infty (H(-t) + H(t))t^{\sigma - \frac{1}{np} - 1} dt < k.$$

Since both

$$\{(H(-t) + H(t))t^{\sigma + \frac{1}{nq} - 1}\}_{n=1}^\infty \quad (t \in (0, 1])$$

and

$$\{(H(-t) + H(t))t^{\sigma - \frac{1}{np} - 1}\}_{n=1}^\infty \quad (t \in (1, \infty))$$

are non-negative and increasing, then by Levi's theorem (cf. [34]), it follows that

$$\begin{aligned} K(\sigma) &= \int_0^1 (H(-t) + H(t))t^{\sigma - 1} dt + \int_1^\infty (H(-t) + H(t))t^{\sigma - 1} dt \\ &= \lim_{n \rightarrow \infty} \left(\int_0^1 (H(-t) + H(t))t^{\sigma + \frac{1}{nq} - 1} dt + \int_1^\infty (H(-t) + H(t))t^{\sigma - \frac{1}{np} - 1} dt \right) \\ &\leq k, \end{aligned}$$

and thus $k = K(\sigma)$ is the best possible constant factor of (88).

The constant factor in (89) is still the best possible. Otherwise, we would reach a contradiction by (92). This completes the proof of the theorem.

Theorem 6. *Replacing $p > 1$ by $0 < p < 1$ in Theorem 1, we obtain the equivalent reverses of (88) and (89). If there exists a constant $\delta_0 > 0$, such that for any $\tilde{\sigma} \in (\sigma - \delta_0, \sigma]$,*

$$K(\tilde{\sigma}) = \int_{-\infty}^\infty H(t)|t|^{\tilde{\sigma} - 1} dt \in \mathbf{R}_+,$$

then the constant factors in the reverses of (88) and (89) are the best possible.

Proof. By Lemma 3 and the reverse of Hölder's inequality, we get the reverses of (88), (89) and (92). Similarly, we can set $g(y)$ as in Theorem 5, and prove that the reverses of (88) and (89) are equivalent.

For $n > \frac{2}{\delta_0|q|}$ ($n \in \mathbf{N}$), we set $f_n(x)$ and $g_n(y)$ as in Theorem 1. If there exists a positive number $k \geq K(\sigma)$, such that the reverse of (88) is valid when replacing $K(\sigma)$ by k , then it follows that

$$\frac{1}{2n}I_n > k \frac{1}{2n}L_n,$$

and

$$\int_0^1 (H(-t) + H(t))t^{\sigma + \frac{1}{nq} - 1} dt + \int_1^\infty (H(-t) + H(t))t^{\sigma - \frac{1}{np} - 1} dt > k. \quad (93)$$

Since

$$\{(H(-t) + H(t))t^{\sigma - \frac{1}{np} - 1}\}_{n=1}^{\infty} \quad (t \in (1, \infty))$$

is still a non-negative and increasing sequence, then by Levi's theorem, it follows that

$$\lim_{n \rightarrow \infty} \int_1^{\infty} (H(-t) + H(t))t^{\sigma - \frac{1}{np} - 1} dt = \int_1^{\infty} (H(-t) + H(t))t^{\sigma - 1} dt.$$

Due to the fact that

$$0 \leq (H(-t) + H(t))t^{\sigma + \frac{1}{nq} - 1} \leq (H(-t) + H(t))t^{(\sigma - \frac{\delta_0}{2}) - 1}$$

($t \in (0, 1]$, $n > \frac{2}{\delta_0|q|}$), and

$$0 \leq \int_0^1 (H(-t) + H(t))t^{(\sigma - \frac{\delta_0}{2}) - 1} dt \leq K(\sigma - \frac{\delta_0}{2}) < \infty,$$

then, by Lebesgue's dominated convergence theorem (cf. [34]), it follows that

$$\lim_{n \rightarrow \infty} \int_0^1 (H(-t) + H(t))t^{\sigma + \frac{1}{nq} - 1} dt = \int_0^1 (H(-t) + H(t))t^{\sigma - 1} dt.$$

In view of the above results and (93), we have

$$\begin{aligned} K(\sigma) &= \int_0^1 (H(-t) + H(t))t^{\sigma - 1} dt + \int_1^{\infty} (H(-t) + H(t))t^{\sigma - 1} dt \\ &= \lim_{n \rightarrow \infty} \left(\int_0^1 (H(-t) + H(t))t^{\sigma + \frac{1}{nq} - 1} dt \right. \\ &\quad \left. + \int_1^{\infty} (H(-t) + H(t))t^{\sigma - \frac{1}{np} - 1} dt \right) \geq k, \end{aligned}$$

and then $k = K(\sigma)$ is the best possible constant factor in the reverse of (88).

Simiraly, we can prove that the constant factor in the reverse of (89) is the best possible by using the reverse of (92). \square

3.3 Yang-Hilbert-Type Integral Inequalities in the Whole Plane with Multi-Variables

Theorem 7. Suppose that $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $H(t) \geq 0$, $K(\sigma) \in \mathbf{R}_+$, $\delta \in \{-1, 1\}$, $-\infty \leq a_i < b_i \leq \infty$, $v'_i(s) > 0$ ($s \in (a_i, b_i)$), $v_i(a_i^+) = -\infty$, $v_i(b_i^-) = \infty$ ($i = 1, 2$). If $f(x), g(y) \geq 0$, such that

$$0 < \int_{a_1}^{b_1} \frac{|v_1(x)|^{p(1-\delta\sigma)-1}}{(v'_1(x))^{p-1}} f^p(x) dx < \infty$$

and

$$0 < \int_{a_2}^{b_2} \frac{|v_2(y)|^{q(1-\sigma)-1}}{(v_2'(y))^{q-1}} g^q(y) dy < \infty,$$

then we have the following equivalent inequalities:

$$\begin{aligned} & \int_{a_2}^{b_2} \int_{a_1}^{b_1} H(v_1^\delta(x)v_2(y))f(x)g(y)dx dy \\ & < K(\sigma) \left[\int_{a_1}^{b_1} \frac{|v_1(x)|^{p(1-\delta\sigma)-1}}{(v_1'(x))^{p-1}} f^p(x) dx \right]^{\frac{1}{p}} \\ & \quad \times \left[\int_{a_2}^{b_2} \frac{|v_2(y)|^{q(1-\sigma)-1}}{(v_2'(y))^{q-1}} g^q(y) dy \right]^{\frac{1}{q}}, \end{aligned} \tag{94}$$

$$\begin{aligned} & \int_{a_2}^{b_2} \frac{v_2'(y)}{|v_2(y)|^{1-p\sigma}} \left(\int_{a_1}^{b_1} H(v_1^\delta(x)v_2(y))f(x) dx \right)^p dy \\ & < K^p(\sigma) \int_{a_1}^{b_1} \frac{|v_1(x)|^{p(1-\delta\sigma)-1}}{(v_1'(x))^{p-1}} f^p(x) dx, \end{aligned} \tag{95}$$

where the constant factors $K(\sigma)$ and $K^p(\sigma)$ are the best possible.

Proof. Setting $x = v_1(s)$, $y = v_2(t)$ in (88), we get $dx = v_1'(s)ds$, $dy = v_2'(t)dt$, and

$$\begin{aligned} I &= \int_{a_2}^{b_2} \int_{a_1}^{b_1} H(v_1^\delta(s)v_2(t))f(v_1(s))g(v_2(t))v_1'(s)v_2'(t)ds dt \\ &= \int_{a_2}^{b_2} \int_{a_1}^{b_1} H(v_1^\delta(s)v_2(t))(f(v_1(s))v_1'(s))(g(v_2(t))v_2'(t))ds dt, \\ I_1 &:= \int_{-\infty}^{\infty} |x|^{p(1-\delta\sigma)-1} f^p(x) dx = \int_{a_1}^{b_1} |v_1(s)|^{p(1-\delta\sigma)-1} f^p(v_1(s))v_1'(s) ds, \\ I_2 &:= \int_{-\infty}^{\infty} y^{q(1-\sigma)-1} g^q(y) dy = \int_{a_2}^{b_2} |v_2(t)|^{q(1-\sigma)-1} g^q(v_2(t))v_2'(t) dt. \end{aligned}$$

If we set

$$F(s) = f(v_1(s))v_1'(s) \text{ and } G(t) = g(v_2(t))v_2'(t),$$

we obtain

$$f^p(v_1(s)) = (v_1'(s))^{-p} F^p(s), g^q(v_2(t)) = (v_2'(t))^{-q} G^q(t),$$

and then it follows that

$$I = \int_{a_2}^{b_2} \int_{a_1}^{b_1} H(v_1^\delta(s)v_2(t))F(s)G(t)ds dt,$$

$$I_1 = \int_{a_1}^{b_1} \frac{|v_1(s)|^{p(1-\delta\sigma)-1}}{(v_1'(s))^{p-1}} F^p(s) ds,$$

$$I_2 = \int_{a_2}^{b_2} \frac{|v_2(t)|^{q(1-\sigma)-1}}{(v_2'(t))^{q-1}} G^q(t) dt.$$

Substitution of the above results to (88), with

$$s = x, t = y, F(s) = f(x), \text{ and } G(t) = g(y),$$

we obtain (94). Similarly, we derive (95). On the other hand, if we set

$$v_1(x) = x, v_2(y) = y, a_i = -\infty, b_i = \infty$$

in (94), we get (88). Hence, the inequalities (94) and (88) are equivalent. It is evident that the inequalities (95) and (89) are equivalent. Hence, the inequalities (94) and (95) are equivalent. Since the constant factors in (88) and (89) are the best possible, it follows that the constant factors in (94) and (95) are also the best possible. This completes the proof of the theorem.

Theorem 8. Replacing $p > 1$ by $0 < p < 1$ in Theorem 7, we obtain the equivalent reverses of (94) and (95). If there exists a constant $\delta_0 > 0$, such that for any $\tilde{\sigma} \in (\sigma - \delta_0, \sigma]$,

$$K(\tilde{\sigma}) = \int_{-\infty}^{\infty} H(t) |t|^{\tilde{\sigma}-1} dt \in \mathbf{R}_+,$$

then the constant factors in the reverses of (94) and (95) are the best possible.

Remark 6. We list the following $v_i(s)$ ($i = 1, 2$) that satisfy the conditions of Theorem 7 and Theorem 8:

- (a) $v_i(s) = s^\gamma$, $s \in (-\infty, \infty)$ ($\gamma \in \{a; a = \frac{1}{2k-1}, 2k+1 \ (k \in \mathbf{N})\}$), satisfying $v_i'(s) = \gamma s^{\gamma-1} > 0$;
- (b) $v_i(s) = \tan^\gamma s$, $s \in (-\frac{\pi}{2}, \frac{\pi}{2})$ ($\gamma \in \{a; a = \frac{1}{2k-1}, 2k+1 \ (k \in \mathbf{N})\}$), satisfying $v_i'(s) = \gamma \tan^{\gamma-1} s \sec^2 s > 0$;
- (c) $v_i(s) = \ln^\gamma s$, $s \in (0, \infty)$ ($\gamma \in \{a; a = \frac{1}{2k-1}, 2k+1 \ (k \in \mathbf{N})\}$), satisfying $v_i'(s) = \frac{\gamma}{s} \ln^{\gamma-1} s > 0$;
- (d) $v_i(s) = (e^{|s|} - 1) \operatorname{sgn}(s)$, $s \in (-\infty, \infty)$, satisfying $v_i'(s) = e^{|s|} > 0$.

Definition 10. If $\lambda \in \mathbf{R}$, $K_\lambda(x, y)$ is a non-negative measurable function in \mathbf{R}^2 , satisfying

$$K_\lambda(tx, ty) = |t|^{-\lambda} K_\lambda(x, y),$$

for any $t \in \mathbf{R} \setminus \{0\}$, $x, y \in \mathbf{R}$, then $K_\lambda(x, y)$ is said to be the homogeneous function of degree $-\lambda$ in \mathbf{R}^2 .

In particular, by Theorem 7 and Theorem 8, with $\delta = 1$, we obtain the following integral inequalities with the non-homogeneous kernel:

Corollary 12. Suppose that $p > 1, \frac{1}{p} + \frac{1}{q} = 1, \sigma \in \mathbf{R}, H(t) \geq 0,$

$$K(\sigma) = \int_{-\infty}^{\infty} H(t)|t|^{\sigma-1} dt \in \mathbf{R}_+,$$

$-\infty \leq a_i < b_i \leq \infty, v'_i(s) > 0 (s \in (a_i, b_i)), v_i(a_i^+) = -\infty, v_i(b_i^-) = \infty (i = 1, 2).$ If $f(x), g(y) \geq 0,$ such that

$$0 < \int_{a_1}^{b_1} \frac{|v_1(x)|^{p(1-\sigma)-1}}{(v'_1(x))^{p-1}} f^p(x) dx < \infty$$

and

$$0 < \int_{a_2}^{b_2} \frac{|v_2(y)|^{q(1-\sigma)-1}}{(v'_2(y))^{q-1}} g^q(y) dy < \infty,$$

then we have the following equivalent inequalities:

$$\begin{aligned} & \int_{a_2}^{b_2} \int_{a_1}^{b_1} H(v_1(x)v_2(y)) f(x)g(y) dx dy \\ & < K(\sigma) \left[\int_{a_1}^{b_1} \frac{|v_1(x)|^{p(1-\sigma)-1}}{(v'_1(x))^{p-1}} f^p(x) dx \right]^{\frac{1}{p}} \\ & \quad \times \left[\int_{a_2}^{b_2} \frac{|v_2(y)|^{q(1-\sigma)-1}}{(v'_2(y))^{q-1}} g^q(y) dy \right]^{\frac{1}{q}}, \end{aligned} \tag{96}$$

$$\begin{aligned} & \int_{a_2}^{b_2} \frac{v'_2(y)}{|v_2(y)|^{1-p\sigma}} \left(\int_{a_1}^{b_1} H(v_1(x)v_2(y)) f(x) dx \right)^p dy \\ & < K^p(\sigma) \int_{a_1}^{b_1} \frac{|v_1(x)|^{p(1-\sigma)-1}}{(v'_1(x))^{p-1}} f^p(x) dx, \end{aligned} \tag{97}$$

where the constant factors $K(\sigma)$ and $K^p(\sigma)$ are the best possible.

Replacing $p > 1$ by $0 < p < 1$ in Corollary 12, we derive the equivalent reverses of (96) and (97). If there exists a constant $\delta_0 > 0,$ such that for any $\tilde{\sigma} \in (\sigma - \delta_0, \sigma],$

$$K(\tilde{\sigma}) = \int_{-\infty}^{\infty} H(t)|t|^{\tilde{\sigma}-1} dt \in \mathbf{R}_+,$$

then the constant factors in the reverses of (96) and (97) are the best possible.

In particular, for $\delta = -1$ in Theorem 7 and Theorem 8, setting $H(t) = K_\lambda(1, t)$ (cf. Definition 10), we get

$$H\left(\frac{v_2(y)}{v_1(x)}\right) = K_\lambda\left(1, \frac{v_2(y)}{v_1(x)}\right) = |v_1(x)|^\lambda K_\lambda(v_1(x), v_2(y)).$$

Replacing $f(x)$ by $|v_1(x)|^{-\lambda} f(x)$, it follows that $|v_1(x)|^{p(1+\sigma)-1} f^p(x)$ is replaced by

$$|v_1(x)|^{p(1+\sigma)-1} [|v_1(x)|^{-\lambda} f(x)]^p = |v_1(x)|^{p(1-\mu)-1} f^p(x),$$

and we have the following integral inequalities with the homogeneous kernel:

Corollary 13. *Let $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $\mu, \sigma \in \mathbf{R}$, $\mu + \sigma = \lambda$, $K_\lambda(x, y)$ is a homogeneous function in \mathbf{R}^2 of degree $-\lambda$,*

$$K_\lambda(\sigma) := \int_{-\infty}^{\infty} K_\lambda(1, t) |t|^{\sigma-1} dt \in \mathbf{R}_+,$$

$0 \leq a_i < b_i \leq \infty$, $v'_i(s) > 0$ ($s \in (a_i, b_i)$), $v_i(a_i^+) = -\infty$, $v_i(b_i^-) = \infty$ ($i = 1, 2$). If $f(x), g(y) \geq 0$, such that

$$0 < \int_{a_1}^{b_1} \frac{|v_1(x)|^{p(1-\mu)-1}}{(v'_1(x))^{p-1}} f^p(x) dx < \infty$$

and

$$0 < \int_{a_2}^{b_2} \frac{|v_2(y)|^{q(1-\sigma)-1}}{(v'_2(y))^{q-1}} g^q(y) dy < \infty,$$

then we have the following equivalent inequalities:

$$\begin{aligned} & \int_{a_2}^{b_2} \int_{a_1}^{b_1} K_\lambda(v_1(x), v_2(y)) f(x) g(y) dx dy \\ & < K_\lambda(\sigma) \left[\int_{a_1}^{b_1} \frac{|v_1(x)|^{p(1-\mu)-1}}{(v'_1(x))^{p-1}} f^p(x) dx \right]^{\frac{1}{p}} \\ & \quad \times \left[\int_{a_2}^{b_2} \frac{|v_2(y)|^{q(1-\sigma)-1}}{(v'_2(y))^{q-1}} g^q(y) dy \right]^{\frac{1}{q}}, \end{aligned} \quad (98)$$

$$\begin{aligned} & \int_{a_2}^{b_2} \frac{v'_2(y)}{|v_2(y)|^{1-p\sigma}} \left(\int_{a_1}^{b_1} K_\lambda(v_1(x), v_2(y)) f(x) dx \right)^p dy \\ & < K_\lambda^p(\sigma) \int_{a_1}^{b_1} \frac{|v_1(x)|^{p(1-\mu)-1}}{(v'_1(x))^{p-1}} f^p(x) dx, \end{aligned} \quad (99)$$

where the constant factors $K_\lambda(\sigma)$ and $K_\lambda^p(\sigma)$ are the best possible.

Replacing $p > 1$ by $0 < p < 1$ in the above cases, we obtain the equivalent reverses of (98) and (99). If there exists a constant $\delta_0 > 0$, such that for any $\tilde{\sigma} \in (\sigma - \delta_0, \sigma]$,

$$K_\lambda(\tilde{\sigma}) = \int_{-\infty}^{\infty} K_\lambda(1, t) |t|^{\tilde{\sigma}-1} dt \in \mathbf{R}_+,$$

then the constant factors in the reverses of (98) and (99) are the best possible.

Setting $a_i = -\infty, b_i = \infty (i = 1, 2), v_1(x) = x, v_2(y) = y$ in Corollary 13, we obtain the following corollary:

Corollary 14. Let $p > 1, \frac{1}{p} + \frac{1}{q} = 1, \mu, \sigma \in \mathbf{R}, \mu + \sigma = \lambda, K_\lambda(x, y)$ is a homogeneous function in \mathbf{R}^2 of degree $-\lambda$,

$$K_\lambda(\sigma) = \int_{-\infty}^{\infty} K_\lambda(1, t)t^{\sigma-1} dt \in \mathbf{R}_+.$$

If $f(x), g(y) \geq 0$,

$$0 < \int_0^\infty x^{p(1-\mu)-1} f^p(x) dx < \infty$$

and

$$0 < \int_0^\infty y^{q(1-\sigma)-1} g^q(y) dy < \infty,$$

then we have the following equivalent inequalities:

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} K_\lambda(x, y) f(x) g(y) dx dy < K_\lambda(\sigma) \left[\int_{-\infty}^{\infty} x^{p(1-\mu)-1} f^p(x) dx \right]^{\frac{1}{p}} \left[\int_{-\infty}^{\infty} y^{q(1-\sigma)-1} g^q(y) dy \right]^{\frac{1}{q}}, \quad (100)$$

$$\int_{-\infty}^{\infty} y^{p\sigma-1} \left(\int_{-\infty}^{\infty} K_\lambda(x, y) f(x) dx \right)^p dy < K_\lambda^p(\sigma) \int_0^\infty x^{p(1-\mu)-1} f^p(x) dx, \quad (101)$$

where the constant factors $K_\lambda(\sigma)$ and $K_\lambda^p(\sigma)$ are the best possible.

Replacing $p > 1$ by $0 < p < 1$ in the above cases, we get the equivalent reverses of (100) and (101). If there exists a constant $\delta_0 > 0$, such that for any $\tilde{\sigma} \in (\sigma - \delta_0, \sigma]$,

$$K_\lambda(\tilde{\sigma}) = \int_{-\infty}^{\infty} K_\lambda(1, t)|t|^{\tilde{\sigma}-1} dt \in \mathbf{R}_+,$$

then the constant factors in the reverses of (100) and (101) are the best possible.

Remark 7. It is evident that (90) and (100) are equivalent for $H(t) = K_\lambda(1, t)$. The same holds for (91) and (101).

3.4 Hardy-Type Integral Inequalities in the Whole Plane

In the following two sections, if the constant factors in the inequalities (operator inequalities) are related to $K^{(1)}(\sigma)$ (or $K_\lambda^{(1)}(\sigma)$), then we shall call them Hardy-type inequalities (operator) of the first kind; if the constant factors in the inequalities (operator inequalities) are related to $K^{(2)}(\sigma)$ (or $K_\lambda^{(2)}(\sigma)$), then we shall call them Hardy-type inequalities (operator) of the second kind.

If $H(t) = 0$ ($|t| > 1$), then $H(xy) = 0$ ($|x| > \frac{1}{|y|} > 0$), and

$$K(\sigma) = \int_{-\infty}^{\infty} H(t)|t|^{\sigma-1} dt = \int_{-1}^1 H(t)|t|^{\sigma-1} dt.$$

Set

$$K^{(1)}(\sigma) := \int_{-1}^1 H(t)|t|^{\sigma-1} dt = \int_0^1 (H(-t) + H(t))t^{\sigma-1} dt. \quad (102)$$

Then, by Theorem 7 and Theorem 8 ($\delta = 1$), we have the following Hardy-type integral inequalities of the first kind, with non-homogeneous kernel in the whole plane:

Corollary 15. *Suppose that $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $H(t) \geq 0$, $\sigma \in \mathbf{R}$,*

$$K^{(1)}(\sigma) = \int_{-1}^1 H(t)|t|^{\sigma-1} dt \in \mathbf{R}_+.$$

If $f(x), g(y) \geq 0$,

$$0 < \int_{-\infty}^{\infty} |x|^{p(1-\sigma)-1} f^p(x) dx < \infty$$

and

$$0 < \int_{-\infty}^{\infty} |y|^{q(1-\sigma)-1} g^q(y) dy < \infty,$$

then we have the following equivalent inequalities:

$$\begin{aligned} \int_{-\infty}^{\infty} \left(\int_{-\frac{1}{|y|}}^{\frac{1}{|y|}} H(xy) f(x) dx \right) g(y) dy &= \int_0^{\infty} \left(\int_{-\frac{1}{|x|}}^{\frac{1}{|x|}} H(xy) g(y) dy \right) f(x) dx \\ &< K^{(1)}(\sigma) \left[\int_{-\infty}^{\infty} |x|^{p(1-\sigma)-1} f^p(x) dx \right]^{\frac{1}{p}} \left[\int_{-\infty}^{\infty} |y|^{q(1-\sigma)-1} g^q(y) dy \right]^{\frac{1}{q}}, \end{aligned} \quad (103)$$

$$\int_{-\infty}^{\infty} |y|^{p\sigma-1} \left(\int_{-\frac{1}{|y|}}^{\frac{1}{|y|}} H(xy) f(x) dx \right)^p dy < (K^{(1)}(\sigma))^p \int_{-\infty}^{\infty} |x|^{p(1-\sigma)-1} f^p(x) dx; \quad (104)$$

where the constant factors $K^{(1)}(\sigma)$ and $(K^{(1)}(\sigma))^p$ are the best possible.

Replacing $p > 1$ by $0 < p < 1$ in the above inequalities, we obtain the equivalent reverses of (103) and (104). If there exists a constant $\delta_0 > 0$, such that for any $\tilde{\sigma} \in (\sigma - \delta_0, \sigma]$,

$$K^{(1)}(\tilde{\sigma}) = \int_{-1}^1 H(t)|t|^{\tilde{\sigma}-1} dt \in \mathbf{R}_+,$$

then the constant factors in the reverses of (103) and (104) are the best possible.

If we have $H(t) = 0$ ($|t| > 1$) in Corollary 12, then

$$H(v_1(x)v_2(y)) = 0(|v_1(x)| > \frac{1}{|v_2(y)|} > 0),$$

and thus we derive the following general results:

Corollary 16. *Suppose that $p > 1, \frac{1}{p} + \frac{1}{q} = 1, H(t) \geq 0, \sigma \in \mathbf{R}$,*

$$K^{(1)}(\sigma) = \int_{-1}^1 H(t)|t|^{\sigma-1} dt \in \mathbf{R}_+,$$

$-\infty \leq a_i < b_i \leq \infty, v'_i(s) > 0 (s \in (a_i, b_i)), v_i(a_i^+) = -\infty, v_i(b_i^-) = \infty (i = 1, 2). If f(x), g(y) \geq 0, such that$

$$0 < \int_{a_1}^{b_1} \frac{|v_1(x)|^{p(1-\sigma)-1}}{(v'_1(x))^{p-1}} f^p(x) dx < \infty$$

and

$$0 < \int_{a_2}^{b_2} \frac{|v_2(y)|^{q(1-\sigma)-1}}{(v'_2(y))^{q-1}} g^q(y) dy < \infty,$$

then we obtain the following equivalent inequalities:

$$\begin{aligned} & \int_{a_2}^{b_2} \left(\int_{v_1^{-1}(\frac{-1}{|v_2(y)|})}^{v_1^{-1}(\frac{1}{|v_2(y)|})} H(v_1(x)v_2(y))f(x)dx \right) g(y)dy \\ & < K^{(1)}(\sigma) \left[\int_{a_1}^{b_1} \frac{|v_1(x)|^{p(1-\sigma)-1}}{(v'_1(x))^{p-1}} f^p(x) dx \right]^{\frac{1}{p}} \\ & \quad \times \left[\int_{a_2}^{b_2} \frac{|v_2(y)|^{q(1-\sigma)-1}}{(v'_2(y))^{q-1}} g^q(y) dy \right]^{\frac{1}{q}}, \end{aligned} \tag{105}$$

$$\begin{aligned} & \int_{a_2}^{b_2} \frac{v'_2(y)}{|v_2(y)|^{1-p\sigma}} \left(\int_{v_1^{-1}(\frac{-1}{|v_2(y)|})}^{v_1^{-1}(\frac{1}{|v_2(y)|})} H(v_1(x)v_2(y))f(x)dx \right)^p dy \\ & < (K^{(1)}(\sigma))^p \int_{a_1}^{b_1} \frac{|v_1(x)|^{p(1-\sigma)-1}}{(v'_1(x))^{p-1}} f^p(x) dx, \end{aligned} \tag{106}$$

where the constant factors $K^{(1)}(\sigma)$ and $(K^{(1)}(\sigma))^p$ are the best possible.

Replacing $p > 1$ by $0 < p < 1$ in the above inequalities, we get the equivalent reverses of (105) and (106). If there exists a constant $\delta_0 > 0$, such that for any $\tilde{\sigma} \in (\sigma - \delta_0, \sigma]$,

$$K^{(1)}(\tilde{\sigma}) = \int_{-1}^1 H(t)|t|^{\tilde{\sigma}-1} dt \in \mathbf{R}_+,$$

then the constant factors in the reverses of (105) and (106) are the best possible.

If $H(t) = 0 (0 < |t| < 1)$, then $H(xy) = 0 (0 < |x| < \frac{1}{|y|})$, and

$$K(\sigma) = \int_{-\infty}^{\infty} H(t)|t|^{\sigma-1} dt = \int_{-\infty}^{-1} H(t)(-t)^{\sigma-1} dt + \int_1^{\infty} H(t)t^{\sigma-1} dt$$

$$= \int_1^{\infty} (H(-t) + H(t))t^{\sigma-1} dt. \quad (107)$$

If we set

$$E_y := \left\{ x \in \mathbf{R}; x \geq \frac{1}{|y|}, \text{ or } x \leq \frac{-1}{|y|} \right\},$$

and

$$K^{(2)}(\sigma) := \int_{E_1} H(t)|t|^{\sigma-1} dt = \int_1^{\infty} (H(-t) + H(t))t^{\sigma-1} dt,$$

then, by Theorem 7 and Theorem 8 ($\delta = 1$), we obtain the following Hardy-type integral inequalities of the second kind with the non-homogeneous kernel in the whole plane:

Corollary 17. *Let $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $H(t) \geq 0$, $\sigma \in \mathbf{R}$,*

$$K^{(2)}(\sigma) = \int_1^{\infty} (H(-t) + H(t))t^{\sigma-1} dt \in \mathbf{R}_+.$$

If $f(x), g(y) \geq 0$,

$$0 < \int_{-\infty}^{\infty} |x|^{p(1-\sigma)-1} f^p(x) dx < \infty$$

and

$$0 < \int_{-\infty}^{\infty} |y|^{q(1-\sigma)-1} g^q(y) dy < \infty,$$

then we have the following equivalent inequalities:

$$\begin{aligned} & \int_{-\infty}^{\infty} \left(\int_{E_y} H(xy) f(x) dx \right) g(y) dy \\ & < K^{(2)}(\sigma) \left[\int_{-\infty}^{\infty} |x|^{p(1-\sigma)-1} f^p(x) dx \right]^{\frac{1}{p}} \left[\int_{-\infty}^{\infty} |y|^{q(1-\sigma)-1} g^q(y) dy \right]^{\frac{1}{q}}, \end{aligned} \quad (108)$$

$$\int_{-\infty}^{\infty} |y|^{p\sigma-1} \left(\int_{E_y} H(xy) f(x) dx \right)^p dy < (K^{(2)}(\sigma))^p \int_{-\infty}^{\infty} |x|^{p(1-\sigma)-1} f^p(x) dx; \quad (109)$$

where the constant factors $K^{(2)}(\sigma)$ and $(K^{(2)}(\sigma))^p$ are the best possible.

Replacing $p > 1$ by $0 < p < 1$ in the above inequalities, we get the equivalent reverses of (108) and (109). If there exists a constant $\delta_0 > 0$, such that for any $\tilde{\sigma} \in (\sigma - \delta_0, \sigma]$,

$$K^{(2)}(\tilde{\sigma}) = \int_1^{\infty} (H(-t) + H(t))t^{\tilde{\sigma}-1} dt \in \mathbf{R}_+,$$

then the constant factors in the reverses of (108) and (109) are the best possible.

If we have $H(t) = 0$ ($0 < |t| < 1$) in Corollary 12, then

$$H(v_1(x)v_2(y)) = 0 \quad (0 < |v_1(x)| < \frac{1}{|v_2(y)|}).$$

Setting

$$\tilde{E}_y := \{x \in (a_1, b_1); x \geq v_1^{-1}\left(\frac{1}{|v_2(y)|}\right) \text{ or } x \leq v_1^{-1}\left(\frac{-1}{|v_2(y)|}\right)\},$$

we obtain the following general results:

Corollary 18. Let $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $H(t) \geq 0$, $\sigma \in \mathbf{R}$,

$$K^{(2)}(\sigma) = \int_1^\infty (H(-t) + H(t))t^{\sigma-1} dt \in \mathbf{R}_+,$$

$-\infty \leq a_i < b_i \leq \infty$, $v_i'(s) > 0$ ($s \in (a_i, b_i)$), $v_i(a_i^+) = -\infty$, $v_i(b_i^-) = \infty$ ($i = 1, 2$). If $f(x), g(y) \geq 0$, such that

$$0 < \int_{a_1}^{b_1} \frac{|v_1(x)|^{p(1-\sigma)-1}}{(v_1'(x))^{p-1}} f^p(x) dx < \infty$$

and

$$0 < \int_{a_2}^{b_2} \frac{|v_2(y)|^{q(1-\sigma)-1}}{(v_2'(y))^{q-1}} g^q(y) dy < \infty,$$

then we have the following equivalent inequalities:

$$\begin{aligned} & \int_{a_2}^{b_2} \left(\int_{\tilde{E}_y} H(v_1(x)v_2(y)) f(x) dx \right) g(y) dy \\ & < K^{(2)}(\sigma) \left[\int_{a_1}^{b_1} \frac{|v_1(x)|^{p(1-\sigma)-1}}{(v_1'(x))^{p-1}} f^p(x) dx \right]^{\frac{1}{p}} \\ & \quad \times \left[\int_{a_2}^{b_2} \frac{|v_2(y)|^{q(1-\sigma)-1}}{(v_2'(y))^{q-1}} g^q(y) dy \right]^{\frac{1}{q}}, \end{aligned} \quad (110)$$

$$\begin{aligned} & \int_{a_2}^{b_2} \frac{v_2'(y)}{|v_2(y)|^{1-p\sigma}} \left(\int_{\tilde{E}_y} H(v_1(x)v_2(y)) f(x) dx \right)^p dy \\ & < (K^{(2)}(\sigma))^p \int_{a_1}^{b_1} \frac{|v_1(x)|^{p(1-\sigma)-1}}{(v_1'(x))^{p-1}} f^p(x) dx, \end{aligned} \quad (111)$$

where the constant factors $K^{(2)}(\sigma)$ and $(K^{(2)}(\sigma))^p$ are the best possible.

Replacing $p > 1$ by $0 < p < 1$ in the above inequalities, we get the equivalent reverses of (110) and (111). If there exists a constant $\delta_0 > 0$, such that for any $\tilde{\sigma} \in (\sigma - \delta_0, \sigma]$,

$$K^{(2)}(\tilde{\sigma}) = \int_1^\infty (H(-t) + h(t))t^{\tilde{\sigma}-1} dt \in \mathbf{R}_+,$$

then the constant factors in the reverses of (110) and (111) are the best possible.

Similarly, if $K_\lambda(1, t) = 0$ ($|t| > 1$), then

$$K_\lambda(x, y) = |x|^{-\lambda} K_\lambda\left(1, \frac{y}{x}\right) = 0 \quad (|y| > |x|).$$

By Corollary 14, setting

$$F_y := \{x \in \mathbf{R}; x \geq |y| \text{ or } x \leq -|y|\},$$

we obtain the following Hardy-type integral inequalities of the first kind, with the homogeneous kernel in the whole plane:

Corollary 19. Let $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $\mu, \sigma \in \mathbf{R}$, $\mu + \sigma = \lambda$, $K_\lambda(x, y)$ is a homogeneous function of degree $-\lambda$ in \mathbf{R}^2 ,

$$K_\lambda^{(1)}(\sigma) = \int_{-1}^1 K_\lambda(1, t) |t|^{\sigma-1} dt \in \mathbf{R}_+.$$

If $f(x), g(y) \geq 0$, such that

$$0 < \int_{-\infty}^{\infty} |x|^{p(1-\mu)-1} f^p(x) dx < \infty$$

and

$$0 < \int_{-\infty}^{\infty} |y|^{q(1-\sigma)-1} g^q(y) dy < \infty,$$

then we have the following equivalent inequalities:

$$\begin{aligned} & \int_{-\infty}^{\infty} \left(\int_{F_y} K_\lambda(x, y) f(x) dx \right) g(y) dy \\ & < K_\lambda^{(1)}(\sigma) \left[\int_{-\infty}^{\infty} |x|^{p(1-\mu)-1} f^p(x) dx \right]^{\frac{1}{p}} \left[\int_{-\infty}^{\infty} |y|^{q(1-\sigma)-1} g^q(y) dy \right]^{\frac{1}{q}}, \end{aligned} \quad (112)$$

$$\int_{-\infty}^{\infty} |y|^{p\sigma-1} \left(\int_{F_y} K_\lambda(x, y) f(x) dx \right)^p dy < (K_\lambda^{(1)}(\sigma))^p \int_{-\infty}^{\infty} |x|^{p(1-\mu)-1} f^p(x) dx, \quad (113)$$

where the constant factors $K_\lambda^{(1)}(\sigma)$ and $(K_\lambda^{(1)}(\sigma))^p$ are the best possible.

Replacing $p > 1$ by $0 < p < 1$ in the above inequalities, we get the equivalent reverses of (112) and (113). If there exists a constant $\delta_0 > 0$, such that for any $\tilde{\sigma} \in (\sigma - \delta_0, \sigma]$,

$$K_\lambda^{(1)}(\tilde{\sigma}) = \int_{-1}^1 K_\lambda(1, t) |t|^{\tilde{\sigma}-1} dt \in \mathbf{R}_+,$$

then the constant factors in the reverses of (112) and (113) are the best possible.

If $K_\lambda(1, t) = 0$ ($|t| > 1$) in Corollary 12, then

$$K_\lambda(v_1(x), v_2(y)) = 0 \quad (0 < |v_1(x)| < |v_2(y)|).$$

Setting

$$\tilde{F}_y := \{x \in (a_1, b_1); x \geq v_1^{-1}(|v_2(y)|) \text{ or } x \leq v_1^{-1}(-|v_2(y)|)\},$$

we obtain the following general results:

Corollary 20. Let $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $\sigma, \mu \in \mathbf{R}$, $\sigma + \mu = \lambda$, $K_\lambda(x, y)$ is a homogeneous function of degree $-\lambda$ in \mathbf{R}^2 ,

$$K_\lambda^{(1)}(\sigma) = \int_{-1}^1 K_\lambda(1, t)|t|^{\sigma-1} dt \in \mathbf{R}_+,$$

$-\infty \leq a_i < b_i \leq \infty$, $v_i'(s) > 0$ ($s \in (a_i, b_i)$), $v_i(a_i^+) = -\infty$, $v_i(b_i^-) = \infty$ ($i = 1, 2$). If $f(x), g(y) \geq 0$, such that

$$0 < \int_{a_1}^{b_1} \frac{|v_1(x)|^{p(1-\mu)-1}}{(v_1'(x))^{p-1}} f^p(x) dx < \infty$$

and

$$0 < \int_{a_2}^{b_2} \frac{|v_2(y)|^{q(1-\sigma)-1}}{(v_2'(y))^{q-1}} g^q(y) dy < \infty,$$

then we have the following equivalent inequalities:

$$\begin{aligned} & \int_{a_2}^{b_2} \left(\int_{\tilde{F}_y} K_\lambda(v_1(x), v_2(y)) f(x) dx \right) g(y) dy \\ & < K_\lambda^{(1)}(\sigma) \left[\int_{a_1}^{b_1} \frac{|v_1(x)|^{p(1-\mu)-1}}{(v_1'(x))^{p-1}} f^p(x) dx \right]^{\frac{1}{p}} \\ & \quad \times \left[\int_{a_2}^{b_2} \frac{|v_2(y)|^{q(1-\sigma)-1}}{(v_2'(y))^{q-1}} g^q(y) dy \right]^{\frac{1}{q}}, \end{aligned} \quad (114)$$

$$\begin{aligned} & \int_{a_2}^{b_2} \frac{v_2'(y)}{|v_2(y)|^{1-p\sigma}} \left(\int_{\tilde{F}_y} K_\lambda(v_1(x), v_2(y)) f(x) dx \right)^p dy \\ & < (K_\lambda^{(1)}(\sigma))^p \int_{a_1}^{b_1} \frac{|v_1(x)|^{p(1-\mu)-1}}{(v_1'(x))^{p-1}} f^p(x) dx, \end{aligned} \quad (115)$$

where the constant factors $K_\lambda^{(1)}(\sigma)$ and $(K_\lambda^{(1)}(\sigma))^p$ are the best possible.

Replacing $p > 1$ by $0 < p < 1$ in the above inequalities, we obtain the equivalent reverses of (114) and (115). If there exists a constant $\delta_0 > 0$, such that for any $\tilde{\sigma} \in (\sigma - \delta_0, \sigma]$,

$$K_\lambda^{(1)}(\tilde{\sigma}) = \int_{-1}^1 K_\lambda(1, t)|t|^{\tilde{\sigma}-1} dt \in \mathbf{R}_+,$$

then the constant factors in the reverses of (114) and (115) are the best possible.

Similarly, if $K_\lambda(1, t) = 0$ ($0 < |t| < 1$) in Corollary 14, then

$$K_\lambda(x, y) = 0 \quad (|x| > |y| > 0).$$

The following Hardy-type integral inequalities of the second kind with the homogeneous kernel, hold true:

Corollary 21. Let $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $\mu, \sigma \in \mathbf{R}$, $\mu + \sigma = \lambda$, $K_\lambda(x, y)$ is a homogeneous function of degree $-\lambda$ in \mathbf{R}^2 ,

$$K_\lambda^{(2)}(\sigma) = \int_1^\infty (K_\lambda(1, -t) + K_\lambda(1, t))t^{\sigma-1} dt \in \mathbf{R}_+.$$

If $f(x), g(y) \geq 0$,

$$0 < \int_{-\infty}^\infty |x|^{p(1-\mu)-1} f^p(x) dx < \infty$$

and

$$0 < \int_{-\infty}^\infty |y|^{q(1-\sigma)-1} g^q(y) dy < \infty,$$

then we have the following equivalent inequalities:

$$\begin{aligned} \int_{-\infty}^\infty \left(\int_{-|y|}^{|y|} K_\lambda(x, y) f(x) dx \right) g(y) dy &= \int_{-\infty}^\infty \left(\int_{-|x|}^{|x|} K_\lambda(x, y) g(y) dy \right) f(x) dx \\ &< K_\lambda^{(2)}(\sigma) \left[\int_{-\infty}^\infty |x|^{p(1-\mu)-1} f^p(x) dx \right]^{\frac{1}{p}} \left[\int_{-\infty}^\infty |y|^{q(1-\sigma)-1} g^q(y) dy \right]^{\frac{1}{q}}, \end{aligned} \quad (116)$$

$$\int_{-\infty}^\infty |y|^{p\sigma-1} \left(\int_{-|y|}^{|y|} K_\lambda(x, y) f(x) dx \right)^p dy < (K_\lambda^{(2)}(\sigma))^p \int_{-\infty}^\infty |x|^{p(1-\mu)-1} f^p(x) dx; \quad (117)$$

where the constant factors $K_\lambda^{(2)}(\sigma)$ and $(K_\lambda^{(2)}(\sigma))^p$ are the best possible.

Replacing $p > 1$ by $0 < p < 1$ in the above inequalities, we obtain the equivalent reverses of (116) and (117). If there exists a constant $\delta_0 > 0$, such that for any $\tilde{\sigma} \in (\sigma - \delta_0, \sigma]$,

$$K_\lambda^{(2)}(\tilde{\sigma}) = \int_1^\infty (K_\lambda(1, -t) + K_\lambda(1, t))t^{\tilde{\sigma}-1} dt \in \mathbf{R}_+,$$

then the constant factors in the reverses of (116) and (117) are the best possible.

If $K_\lambda(1, t) = 0$ ($0 < |t| < 1$) in Corollary 12, then

$$K_\lambda(v_1(x), v_2(y)) = 0 \quad (|v_1(x)| > |v_2(y)| > 0),$$

we have the following general results:

Corollary 22. Let $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $\mu, \sigma \in \mathbf{R}$, $\mu + \sigma = \lambda$, $K_\lambda(x, y)$ is a homogeneous function of degree $-\lambda$ in \mathbf{R}^2 ,

$$K_\lambda^{(2)}(\sigma) = \int_1^\infty (K_\lambda(1, -t) + K_\lambda(1, t))t^{\sigma-1} dt \in \mathbf{R}_+,$$

$-\infty \leq a_i < b_i \leq \infty, v_i'(s) > 0 (s \in (a_i, b_i)), v_i(a_i^+) = -\infty, v_i(b_i^-) = \infty (i = 1, 2)$. If $f(x), g(y) \geq 0$, such that

$$0 < \int_{a_1}^{b_1} \frac{|v_1(x)|^{p(1-\mu)-1}}{(v_1'(x))^{p-1}} f^p(x) dx < \infty$$

and

$$0 < \int_{a_2}^{b_2} \frac{|v_2(y)|^{q(1-\sigma)-1}}{(v_2'(y))^{q-1}} g^q(y) dy < \infty,$$

then we have the following equivalent inequalities:

$$\begin{aligned} & \int_{a_2}^{b_2} \left(\int_{v_1^{-1}(-|v_2(y)|)}^{v_1^{-1}(|v_2(y)|)} K_\lambda(v_1(x), v_2(y)) f(x) dx \right) g(y) dy \\ & < K_\lambda^{(2)}(\sigma) \left[\int_{a_1}^{b_1} \frac{|v_1(x)|^{p(1-\mu)-1}}{(v_1'(x))^{p-1}} f^p(x) dx \right]^{\frac{1}{p}} \\ & \quad \times \left[\int_{a_2}^{b_2} \frac{|v_2(y)|^{q(1-\sigma)-1}}{(v_2'(y))^{q-1}} g^q(y) dy \right]^{\frac{1}{q}}, \end{aligned} \tag{118}$$

$$\begin{aligned} & \int_{a_2}^{b_2} \frac{v_2'(y)}{|v_2(y)|^{1-p\sigma}} \left(\int_{v_1^{-1}(-|v_2(y)|)}^{v_1^{-1}(|v_2(y)|)} K_\lambda(v_1(x), v_2(y)) f(x) dx \right)^p dy \\ & < (K_\lambda^{(2)}(\sigma))^p \int_{a_1}^{b_1} \frac{|v_1(x)|^{p(1-\mu)-1}}{(v_1'(x))^{p-1}} f^p(x) dx, \end{aligned} \tag{119}$$

where the constant factors $K_\lambda^{(2)}(\sigma)$ and $(K_\lambda^{(2)}(\sigma))^p$ are the best possible.

Replacing $p > 1$ by $0 < p < 1$ in the above inequalities, we have the equivalent reverses of (118) and (119). If there exists a constant $\delta_0 > 0$, such that for any $\tilde{\sigma} \in (\sigma - \delta_0, \sigma]$,

$$K_\lambda^{(2)}(\tilde{\sigma}) = \int_1^\infty (K_\lambda(1, -t) + K_\lambda(1, t))t^{\tilde{\sigma}-1} dt \in \mathbf{R}_+,$$

then the constant factors in the reverses of (118) and (119) are the best possible.

3.5 Yang-Hilbert-Type Operators and Hardy-Type Operators in the Whole Plane

Let $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $\mu, \sigma \in \mathbf{R}$, $\mu + \sigma = \lambda$. We define the following functions:

$$\varphi(x) := |x|^{p(1-\sigma)-1}, \psi(y) := |y|^{q(1-\sigma)-1}, \phi(x) := |x|^{p(1-\mu)-1} (x, y \in \mathbf{R}),$$

wherefrom, $\psi^{1-p}(y) = |y|^{p\sigma-1}$.

We define also the following real normed linear space:

$$L_{p,\varphi}(\mathbf{R}) := \left\{ f : \|f\|_{p,\varphi} := \left\{ \int_{-\infty}^{\infty} \varphi(x) |f(x)|^p dx \right\}^{\frac{1}{p}} < \infty \right\},$$

wherefrom,

$$L_{p,\psi^{1-p}}(\mathbf{R}) = \left\{ h : \|h\|_{p,\psi^{1-p}} := \left\{ \int_{-\infty}^{\infty} \psi^{1-p}(y) |h(y)|^p dy \right\}^{\frac{1}{p}} < \infty \right\},$$

$$L_{p,\phi}(\mathbf{R}) = \left\{ g : \|g\|_{p,\phi} := \left\{ \int_{-\infty}^{\infty} \phi(x) |g(x)|^p dx \right\}^{\frac{1}{p}} < \infty \right\}.$$

(a) In view of Theorem 5 ($\delta = 1$), for $f \in L_{p,\varphi}(\mathbf{R})$,

$$H_1(y) := \int_{-\infty}^{\infty} H(xy) |f(x)| dx \quad (y \in \mathbf{R}_+),$$

by (91), we have

$$\|H_1\|_{p,\psi^{1-p}} := \left(\int_{-\infty}^{\infty} \psi^{1-p}(y) H_1^p(y) dy \right)^{\frac{1}{p}} < K(\sigma) \|f\|_{p,\varphi} < \infty. \quad (120)$$

Definition 11. Define Yang-Hilbert-type integral operator with the non-homogeneous kernel in the whole plane

$$T_1 : L_{p,\varphi}(\mathbf{R}) \rightarrow L_{p,\psi^{1-p}}(\mathbf{R})$$

as follows:

For any $f \in L_{p,\varphi}(\mathbf{R})$, there exists a unique representation

$$T_1 f = H_1 \in L_{p,\psi^{1-p}}(\mathbf{R}),$$

satisfying

$$T_1 f(y) = H_1(y),$$

for any $y \in \mathbf{R}$.

In view of (120), it follows that

$$\|T_1 f\|_{p, \psi^{1-p}} = \|H_1\|_{p, \psi^{1-p}} \leq K(\sigma) \|f\|_{p, \varphi}.$$

Therefore, the operator T_1 is bounded and it satisfies the following relation

$$\|T_1\| = \sup_{f(\neq \theta) \in L_{p, \varphi}(\mathbf{R})} \frac{\|T_1 f\|_{p, \psi^{1-p}}}{\|f\|_{p, \varphi}} \leq K(\sigma).$$

Since the constant factor $K(\sigma)$ in (120) is the best possible, we have

$$\|T_1\| = K(\sigma) = \int_{-\infty}^{\infty} H(t) |t|^{\sigma-1} dt. \quad (121)$$

If we define the formal inner product of $T_1 f$ and g as

$$\begin{aligned} (T_1 f, g) &:= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} H(xy) f(x) dx \right) g(y) dy \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} H(xy) f(x) g(y) dx dy, \end{aligned}$$

then we can rewrite (90) and (91) as follows:

$$(T_1 f, g) < \|T_1\| \cdot \|f\|_{p, \varphi} \|g\|_{q, \psi}, \quad \|T_1 f\|_{p, \psi^{1-p}} < \|T_1\| \cdot \|f\|_{p, \varphi}.$$

(b) In view of Corollary 15, for $f \in L_{p, \varphi}(\mathbf{R})$, setting

$$H_1^{(1)}(y) := \int_{\frac{-1}{|y|}}^{\frac{1}{|y|}} H(xy) |f(x)| dx \quad (y \in \mathbf{R} \setminus \{0\}),$$

by (104), we obtain

$$\|H_1^{(1)}\|_{p, \psi^{1-p}} = \left(\int_{-\infty}^{\infty} \psi^{1-p}(y) (H_1^{(1)}(y))^p dy \right)^{\frac{1}{p}} < K^{(1)}(\sigma) \|f\|_{p, \varphi} < \infty. \quad (122)$$

Definition 12. Let us define the Hardy-type integral operator of the first kind, with the non-homogeneous kernel in the whole plane

$$T_1^{(1)} : L_{p, \varphi}(\mathbf{R}) \rightarrow L_{p, \psi^{1-p}}(\mathbf{R})$$

as follows:

For any $f \in L_{p, \varphi}(\mathbf{R})$, there exists a unique representation

$$T_1^{(1)} f = H_1^{(1)} \in L_{p, \psi^{1-p}}(\mathbf{R}),$$

satisfying

$$T_1^{(1)} f(y) = H_1^{(1)}(y),$$

for any $y \in \mathbf{R}$.

In view of (122), it follows that

$$\|T_1^{(1)}f\|_{p,\psi^{1-p}} = \|H_1^{(1)}\|_{p,\psi^{1-p}} \leq K^{(1)}(\sigma)\|f\|_{p,\varphi}.$$

Then, the operator $T_1^{(1)}$ is bounded satisfying

$$\|T_1^{(1)}\| = \sup_{f(\neq\theta)\in L_{p,\varphi}(\mathbf{R})} \frac{\|T_1^{(1)}f\|_{p,\psi^{1-p}}}{\|f\|_{p,\varphi}} \leq K^{(1)}(\sigma).$$

Since the constant factor $K^{(1)}(\sigma)$ in (122) is the best possible, we have

$$\|T_1^{(1)}\| = K^{(1)}(\sigma) = \int_{-1}^1 H(t)|t|^{\sigma-1} dt. \quad (123)$$

Setting the formal inner product of $T_1^{(1)}f$ and g as

$$(T_1^{(1)}f, g) = \int_{-\infty}^{\infty} \left(\int_{\frac{-1}{|y|}}^{\frac{1}{|y|}} H(xy)f(x)dx \right) g(y)dy,$$

we can rewrite (103) and (104) as follows:

$$(T_1^{(1)}f, g) < \|T_1^{(1)}\| \cdot \|f\|_{p,\varphi} \|g\|_{q,\psi}, \quad \|T_1^{(1)}f\|_{p,\psi^{1-p}} < \|T_1^{(1)}\| \cdot \|f\|_{p,\varphi}. \quad (124)$$

(c) In view of Corollary 17, for $f \in L_{p,\varphi}(\mathbf{R})$, setting

$$H_1^{(2)}(y) := \int_{E_y} H(xy)|f(x)|dx \quad (y \in \mathbf{R}),$$

by (109), we have

$$\begin{aligned} \|H_1^{(2)}\|_{p,\psi^{1-p}} &= \left(\int_{-\infty}^{\infty} \psi^{1-p}(y)(H_1^{(2)}(y))^p dy \right)^{\frac{1}{p}} \\ &< K^{(2)}(\sigma)\|f\|_{p,\varphi} < \infty. \end{aligned} \quad (125)$$

Definition 13. Let us define the Hardy-type integral operator of the second kind with the non-homogeneous kernel in the whole plane

$$T_1^{(2)} : L_{p,\varphi}(\mathbf{R}) \rightarrow L_{p,\psi^{1-p}}(\mathbf{R})$$

as follows:

For any $f \in L_{p,\varphi}(\mathbf{R})$, there exists a unique representation

$$T_1^{(2)}f = H_1^{(2)} \in L_{p,\psi^{1-p}}(\mathbf{R}),$$

satisfying

$$T_1^{(2)}f(y) = H_1^{(2)}(y),$$

for any $y \in \mathbf{R}$.

In view of (125), it follows that

$$\|T_1^{(2)}f\|_{p,\psi^{1-p}} = \|H_1^{(2)}\|_{p,\psi^{1-p}} \leq K^{(2)}(\sigma)\|f\|_{p,\phi}.$$

Thus, the operator $T_1^{(2)}$ is bounded satisfying

$$\|T_1^{(2)}\| = \sup_{f(\neq\theta) \in L_{p,\phi}(\mathbf{R})} \frac{\|T_1^{(2)}f\|_{p,\psi^{1-p}}}{\|f\|_{p,\phi}} \leq K^{(2)}(\sigma).$$

Since the constant factor $K^{(2)}(\sigma)$ in (125) is the best possible, we have

$$\|T_1^{(2)}\| = K^{(2)}(\sigma) = \int_1^\infty (H(-t) + H(t))t^{\sigma-1} dt. \quad (126)$$

Setting the formal inner product of $T_1^{(2)}f$ and g as

$$(T_1^{(2)}f, g) = \int_{-\infty}^\infty \left(\int_{E_y} H(xy)f(x)dx \right) g(y)dy,$$

we can rewrite (108) and (109) as follows:

$$(T_1^{(2)}f, g) < \|T_1^{(2)}\| \cdot \|f\|_{p,\phi} \|g\|_{q,\psi}, \quad \|T_1^{(2)}f\|_{p,\psi^{1-p}} < \|T_1^{(2)}\| \cdot \|f\|_{p,\phi}. \quad (127)$$

(d) In view of Corollary 14, for $f \in L_{p,\phi}(\mathbf{R})$,

$$H_2(y) := \int_{-\infty}^\infty K_\lambda(x, y)|f(x)|dx \quad (y \in \mathbf{R}),$$

by (101), we have

$$\begin{aligned} \|H_2\|_{p,\psi^{1-p}} &= \left(\int_{-\infty}^\infty \psi^{1-p}(y)H_2^p(y)dy \right)^{\frac{1}{p}} \\ &< K_\lambda(\sigma)\|f\|_{p,\phi} < \infty. \end{aligned} \quad (128)$$

Definition 14. We define the Yang-Hilbert-type integral operator with the homogeneous kernel in the whole plane

$$T_2 : L_{p,\phi}(\mathbf{R}) \rightarrow L_{p,\psi^{1-p}}(\mathbf{R})$$

as follows:

For any $f \in L_{p,\phi}(\mathbf{R})$, there exists a unique representation

$$T_2f = H_2 \in L_{p,\psi^{1-p}}(\mathbf{R}),$$

satisfying

$$T_2 f(y) = H_2(y),$$

for any $y \in \mathbf{R}$.

By (128), it follows that

$$\|T_2 f\|_{p, \psi^{1-p}} = \|H_2\|_{p, \psi^{1-p}} \leq K_\lambda(\sigma) \|f\|_{p, \phi}.$$

Hence, the operator T_2 is bounded satisfying

$$\|T_2\| = \sup_{f(\neq \theta) \in L_{p, \phi}(\mathbf{R})} \frac{\|T_2 f\|_{p, \psi^{1-p}}}{\|f\|_{p, \phi}} \leq K_\lambda(\sigma).$$

Since the constant factor $K_\lambda(\sigma)$ in (128) is the best possible, we have

$$\|T_2\| = K_\lambda(\sigma) = \int_{-\infty}^{\infty} K_\lambda(1, t) |t|^{\sigma-1} dt. \quad (129)$$

Setting the formal inner product of $T_2 f$ and g as

$$\begin{aligned} (T_2 f, g) &= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} K_\lambda(x, y) f(x) dx \right) g(y) dy \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} K_\lambda(x, y) f(x) g(y) dx dy, \end{aligned}$$

we can rewrite (100) and (101) as follows:

$$(T_2 f, g) < \|T_2\| \cdot \|f\|_{p, \phi} \|g\|_{q, \psi}, \quad \|T_2 f\|_{p, \psi^{1-p}} < \|T_2\| \cdot \|f\|_{p, \phi}.$$

(e) By Corollary 19, for $f \in L_{p, \phi}(\mathbf{R})$,

$$H_2^{(1)}(y) := \int_{F_y} K_\lambda(x, y) |f(x)| dx \quad (y \in \mathbf{R}),$$

combined with (113), we obtain

$$\begin{aligned} \|H_2^{(1)}\|_{p, \psi^{1-p}} &= \left(\int_{-\infty}^{\infty} \psi^{1-p}(y) (H_2^{(1)}(y))^p dy \right)^{\frac{1}{p}} \\ &< K_\lambda^{(1)}(\sigma) \|f\|_{p, \phi} < \infty. \end{aligned} \quad (130)$$

Definition 15. We define the Hardy-type integral operator of the first kind, with the homogeneous kernel in the whole plane

$$T_2^{(1)} : L_{p, \phi}(\mathbf{R}) \rightarrow L_{p, \psi^{1-p}}(\mathbf{R})$$

as follows:

For any $f \in L_{p, \phi}(\mathbf{R})$, there exists a unique representation

$$T_2^{(1)}f = H_2^{(1)} \in L_{p,\psi^{1-p}}(\mathbf{R}),$$

satisfying

$$T_2^{(1)}f(y) = H_2^{(1)}(y),$$

for any $y \in \mathbf{R}$.

In view of (130), it follows that

$$\|T_2^{(1)}f\|_{p,\psi^{1-p}} = \|H_2^{(1)}\|_{p,\psi^{1-p}} \leq K_\lambda^{(1)}(\sigma)\|f\|_{p,\phi}.$$

Therefore, the operator $T_2^{(1)}$ is bounded satisfying

$$\|T_2^{(1)}\| = \sup_{f(\neq\theta) \in L_{p,\phi}(\mathbf{R})} \frac{\|T_2^{(1)}f\|_{p,\psi^{1-p}}}{\|f\|_{p,\phi}} \leq K_\lambda^{(1)}(\sigma).$$

Since the constant factor $K_\lambda^{(1)}(\sigma)$ in (130) is the best possible, we have

$$\|T_2^{(1)}\| = K_\lambda^{(1)}(\sigma) = \int_{-1}^1 K_\lambda(1,t)|t|^{\sigma-1} dt. \quad (131)$$

Setting the formal inner product of $T_2^{(1)}f$ and g as

$$(T_2^{(1)}f, g) = \int_{-\infty}^{\infty} \left(\int_{F_y} K_\lambda(x,y)f(x)dx \right) g(y)dy,$$

we can rewrite (112) and (113) as follows:

$$(T_2^{(1)}f, g) < \|T_2^{(1)}\| \cdot \|f\|_{p,\phi} \|g\|_{q,\psi}, \quad \|T_2^{(1)}f\|_{p,\psi^{1-p}} < \|T_2^{(1)}\| \cdot \|f\|_{p,\phi}.$$

(f) In view of Corollary 21, for $f \in L_{p,\phi}(\mathbf{R}_+)$,

$$H_2^{(2)}(y) := \int_{-|y|}^{|y|} K_\lambda(x,y)|f(x)|dx \quad (y \in \mathbf{R}),$$

by (117), we have

$$\begin{aligned} \|H_2^{(2)}\|_{p,\psi^{1-p}} &: = \left(\int_{-\infty}^{\infty} \psi^{1-p}(y)(H_2^{(2)}(y))^p dy \right)^{\frac{1}{p}} \\ &< K_\lambda^{(2)}(\sigma)\|f\|_{p,\phi} < \infty. \end{aligned} \quad (132)$$

Definition 16. We define the Hardy-type integral operator of the second kind, with the homogeneous kernel in the whole plane

$$T_2^{(2)} : L_{p,\phi}(\mathbf{R}) \rightarrow L_{p,\psi^{1-p}}(\mathbf{R})$$

as follows:

For any $f \in L_{p,\phi}(\mathbf{R})$, there exists a unique representation

$$T_2^{(2)}f = H_2^{(2)} \in L_{p,\psi^{1-p}}(\mathbf{R}),$$

satisfying

$$T_2^{(2)}f(y) = H_2^{(2)}(y),$$

for any $y \in \mathbf{R}$.

By (132), it follows that

$$\|T_2^{(2)}f\|_{p,\psi^{1-p}} = \|H_2^{(2)}\|_{p,\psi^{1-p}} \leq K_\lambda^{(2)}(\sigma)\|f\|_{p,\phi}$$

and then the operator $T_2^{(2)}$ is bounded satisfying

$$\|T_2^{(2)}\| = \sup_{f(\neq\theta) \in L_{p,\phi}(\mathbf{R})} \frac{\|T_2^{(2)}f\|_{p,\psi^{1-p}}}{\|f\|_{p,\phi}} \leq K_\lambda^{(2)}(\sigma).$$

Since the constant factor $K_\lambda^{(2)}(\sigma)$ in (132) is the best possible, we have

$$\|T_2^{(2)}\| = K_\lambda^{(2)}(\sigma) = \int_1^\infty (K_\lambda(1, -t) + K_\lambda(1, t))t^{\sigma-1} dt. \quad (133)$$

Setting the formal inner product of $T_2^{(2)}f$ and g as

$$(T_2^{(2)}f, g) = \int_{-\infty}^\infty \left(\int_{-|y|}^{|y|} K_\lambda(x, y)f(x)dx \right) g(y)dy,$$

we can rewrite (116) and (117) as follows:

$$(T_2^{(2)}f, g) < \|T_2^{(2)}\| \cdot \|f\|_{p,\phi} \|g\|_{q,\psi}, \quad \|T_2^{(2)}f\|_{p,\psi^{1-p}} < \|T_2^{(2)}\| \cdot \|f\|_{p,\phi}.$$

Remark 8. (a) If $K_\lambda(x, y)$ is a symmetric function satisfying $K_\lambda(y, x) = K_\lambda(x, y)$, then by setting

$$H(t) =: K_\lambda(1, t) \arctan |t|^\beta \quad (\beta \in \mathbf{R}),$$

and $\mu = \sigma = \frac{\lambda}{2}$ in (121), we obtain

$$\begin{aligned} \|T_1\| &= \int_{-\infty}^\infty H(t)|t|^{\sigma-1} = \int_{-\infty}^\infty K_\lambda(1, t) \arctan |t|^\beta |t|^{\sigma-1} dt \\ &= \frac{\pi}{4} K_\lambda\left(\frac{\lambda}{2}\right), \end{aligned} \quad (134)$$

where

$$K_\lambda\left(\frac{\lambda}{2}\right) = \int_{-\infty}^\infty K_\lambda(1, t)|t|^{\sigma-1} dt.$$

In fact, we obtain

$$\begin{aligned} & \int_0^\infty K_\lambda(1,t)(\arctan t^\beta)t^{\frac{\lambda}{2}-1} dt \\ &= \int_0^1 K_\lambda(1,t)(\arctan t^\beta)t^{\frac{\lambda}{2}-1} dt + \int_1^\infty K_\lambda(1,u)(\arctan u^\beta)u^{\frac{\lambda}{2}-1} du \\ &= \int_0^1 K_\lambda(1,t)(\arctan t^\beta)t^{\frac{\lambda}{2}-1} dt + \int_0^1 K_\lambda(t,1)(\arctan t^{-\beta})t^{\frac{\lambda}{2}-1} dt \\ &= \int_0^1 K_\lambda(1,t)(\arctan t^\beta + \arctan t^{-\beta})t^{\frac{\lambda}{2}-1} dt \\ &= \frac{\pi}{2} \int_0^1 K_\lambda(1,t)t^{\frac{\lambda}{2}-1} dt = \frac{\pi}{4} \int_0^\infty K_\lambda(1,t)t^{\frac{\lambda}{2}-1} dt. \end{aligned}$$

Similarly, we get

$$\begin{aligned} & \int_{-\infty}^0 K_\lambda(1,t) \arctan(-t)^\beta (-t)^{\sigma-1} dt \\ &= \int_0^\infty K_\lambda(1,-u)(\arctan u^\beta)u^{\sigma-1} du = \frac{\pi}{4} \int_0^\infty K_\lambda(1,-t)t^{\frac{\lambda}{2}-1} dt. \end{aligned}$$

Then we have

$$\|T_1\| = \frac{\pi}{4} \int_0^\infty (K_\lambda(1,-t) + K_\lambda(1,t))t^{\frac{\lambda}{2}-1} dt = \frac{\pi}{4} K_\lambda\left(\frac{\lambda}{2}\right).$$

(b) If we replace $H(t)$ by

$$h(|t|^\gamma + t^\gamma \cos \alpha) (\gamma \in \{b; b = \frac{1}{2k-1}, 2k+1 (k \in \mathbf{N})\}, \alpha \in (0, \pi))$$

in (121), where, $h(t)$ is a non-negative measurable function in \mathbf{R}_+ , satisfying

$$k\left(\frac{\sigma}{\gamma}\right) = \int_0^\infty h(t)t^{\frac{\sigma}{\gamma}-1} dt \in \mathbf{R}_+,$$

it follows that

$$\begin{aligned} \|T_1\| &= \int_{-\infty}^\infty h(|t|^\gamma + t^\gamma \cos \alpha)|t|^{\sigma-1} \\ &= \frac{1}{\gamma 2^{\sigma/\gamma}} \left[\left(\sec \frac{\alpha}{2}\right)^{\frac{2\alpha}{\gamma}} + \left(\csc \frac{\alpha}{2}\right)^{\frac{2\alpha}{\gamma}} \right] k\left(\frac{\sigma}{\gamma}\right), \end{aligned} \tag{135}$$

In particular, setting $h(t) = k_\lambda(1,t)$ ($t \in \mathbf{R}_+$), it follows that

$$\begin{aligned} \|T_1\| &= \int_{-\infty}^\infty k_\lambda(1,|t|^\gamma + t^\gamma \cos \alpha)|t|^{\sigma-1} \\ &= \frac{1}{\gamma 2^{\sigma/\gamma}} \left[\left(\sec \frac{\alpha}{2}\right)^{\frac{2\alpha}{\gamma}} + \left(\csc \frac{\alpha}{2}\right)^{\frac{2\alpha}{\gamma}} \right] k_\lambda\left(\frac{\sigma}{\gamma}\right), \end{aligned} \tag{136}$$

where

$$k_\lambda\left(\frac{\sigma}{\gamma}\right) = \int_0^\infty k_\lambda(1, t)t^{\frac{\sigma}{\gamma}-1} dt \in \mathbf{R}_+.$$

In fact, setting $u = t^\gamma(1 + \cos \alpha)$, we get

$$\begin{aligned} & \int_0^\infty h(|t|^\gamma + t^\gamma \cos \alpha)|t|^{\sigma-1} dt \\ &= \int_0^\infty h(t^\gamma(1 + \cos \alpha))t^{\sigma-1} dt = \frac{1}{\gamma 2^{\sigma/\gamma}} (\sec \frac{\alpha}{2})^{\frac{2\alpha}{\gamma}} k\left(\frac{\sigma}{\gamma}\right); \end{aligned}$$

Moreover, setting $u = -t$, we have

$$\begin{aligned} & \int_{-\infty}^0 h(|t|^\gamma + t^\gamma \cos \alpha)|t|^{\sigma-1} dt = \int_{-\infty}^0 h(-t^\gamma(1 - \cos \alpha))(-t)^{\sigma-1} dt \\ &= \int_0^\infty h(u^\gamma(1 - \cos \alpha))u^{\sigma-1} du = \frac{1}{\gamma 2^{\sigma/\gamma}} (\csc \frac{\alpha}{2})^{\frac{2\alpha}{\gamma}} k\left(\frac{\sigma}{\gamma}\right), \end{aligned}$$

and then (135) follows.

(c) Replacing $K_\lambda(1, t)$ by $k_\lambda(1, |t|^\gamma + t^\gamma \cos \alpha)$ in (134), where, $k_\lambda(x, y)$ is a homogeneous function in \mathbf{R}_+ , satisfying

$$k_\lambda\left(\frac{\sigma}{\gamma}\right) = \int_0^\infty k_\lambda(1, t)t^{\frac{\sigma}{\gamma}-1} dt \in \mathbf{R}_+,$$

in view of (136), we obtain

$$\begin{aligned} \|T_1\| &= \int_{-\infty}^\infty k_\lambda(1, |t|^\gamma + t^\gamma \cos \alpha) \arctan |t|^\beta |t|^{\frac{\lambda}{2}-1} dt \\ &= \frac{\pi}{\gamma 2^{2+\sigma/\gamma}} \left[\left(\sec \frac{\alpha}{2}\right)^{\frac{2\alpha}{\gamma}} + \left(\csc \frac{\alpha}{2}\right)^{\frac{2\alpha}{\gamma}} \right] k_\lambda\left(\frac{\lambda}{2\gamma}\right). \end{aligned} \quad (137)$$

3.6 Some Examples

Example 5. (a) Set

$$H(t) = K_\lambda(1, t) = \frac{1}{|1+t|^\lambda} \quad (\mu, \sigma > 0, \mu + \sigma = \lambda < 1).$$

Then we have the kernels

$$H(xy) = \frac{1}{|1+xy|^\lambda}, \quad K_\lambda(x, y) = \frac{1}{|x+y|^\lambda}$$

and obtain the constant factors

$$\begin{aligned}
K(\sigma) &= K_\lambda(\sigma) = \int_{-\infty}^{\infty} \frac{|t|^{\sigma-1}}{|1+t|^\lambda} dt \\
&= \int_0^{\infty} \frac{t^{\sigma-1}}{|1-t|^\lambda} dt + \int_0^{\infty} \frac{t^{\sigma-1}}{(1+t)^\lambda} dt \\
&= B(1-\lambda, \sigma) + B(1-\lambda, \mu) + B(\mu, \sigma) \in \mathbf{R}_+.
\end{aligned}$$

By (121) and (129), we have (cf. [16])

$$\|T_1\| = \|T_2\| = B(1-\lambda, \sigma) + B(1-\lambda, \mu) + B(\mu, \sigma). \quad (138)$$

(b) Set

$$H(t) = K_\lambda(1, t) = \frac{|\ln |t|^\beta|}{|1+t|^\lambda},$$

where $\beta > -1, \mu, \sigma > 0, \mu + \sigma = \lambda < 1 + \beta$. Then we have the kernels

$$H(xy) = \frac{|\ln |xy|^\beta|}{|1+xy|^\lambda}, \quad K_\lambda(x, y) = \frac{|\ln |x/y|^\beta|}{|x+y|^\lambda}$$

and obtain the constant factors

$$\begin{aligned}
K(\sigma) &= K_\lambda(\sigma) = \int_{-\infty}^{\infty} \frac{|\ln |t|^\beta| |t|^{\sigma-1}}{|1+t|^\lambda} dt \\
&= \int_0^{\infty} \frac{|\ln t^\beta| t^{\sigma-1}}{|1-t|^\lambda} dt + \int_0^{\infty} \frac{|\ln t^\beta| t^{\sigma-1}}{(1+t)^\lambda} dt \\
&= \int_0^1 (-\ln t)^\beta \left[\frac{1}{(1-t)^\lambda} + \frac{1}{(1+t)^\lambda} \right] (t^{\mu-1} + t^{\sigma-1}) dt.
\end{aligned}$$

There exists a constant $\delta_0 > 0$, such that $\sigma > \delta_0, \mu > \delta_0$. Since

$$\lim_{t \rightarrow 0^+} t^{\delta_0} \frac{(-\ln t)^\beta}{(1-t)^\beta} = 0, \quad \lim_{t \rightarrow 1^-} t^{\delta_0} \frac{(-\ln t)^\beta}{(1-t)^\beta} = 1,$$

there exists a constant $L > 0$, such that

$$t^{\delta_0} \frac{(-\ln t)^\beta}{(1-t)^\beta} \leq L \quad (0 < t < 1)$$

and

$$\begin{aligned}
0 < K(\sigma) &= \int_0^1 (-\ln t)^\beta \left[\frac{1}{(1-t)^\lambda} + \frac{1}{(1+t)^\lambda} \right] (t^{\mu-1} + t^{\sigma-1}) dt \\
&\leq 2 \int_0^1 \frac{(-\ln t)^\beta}{(1-t)^\lambda} (t^{\mu-1} + t^{\sigma-1}) dt \leq 2L \int_0^1 \frac{t^{\mu-\delta_0-1} + t^{\sigma-\delta_0-1}}{(1-t)^{\lambda-\beta}} dt \\
&= 2L[B(\beta+1-\lambda, \mu-\delta_0) + B(\beta+1-\lambda, \sigma-\delta_0)] < \infty.
\end{aligned}$$

Therefore, $K(\sigma) = K_\lambda(\sigma) \in \mathbf{R}_+$.

Since

$$\binom{-\lambda}{2k} = \binom{\lambda + 2k - 1}{2k} > 0,$$

then by Lebesgue's term by term theorem, it follows that

$$\begin{aligned} K(\sigma) &= \int_0^1 (-\ln t)^\beta \sum_{k=0}^{\infty} \binom{-\lambda}{2k} [(-1)^k + 1] (t^{k+\mu-1} + t^{k+\sigma-1}) dt \\ &= 2 \int_0^1 (-\ln t)^\beta \sum_{k=0}^{\infty} \binom{-\lambda}{2k} (t^{k+\mu-1} + t^{k+\sigma-1}) dt \\ &= 2 \sum_{k=0}^{\infty} \binom{-\lambda}{2k} \int_0^1 (-\ln t)^\beta (t^{2k+\mu-1} + t^{2k+\sigma-1}) dt \\ &= 2\Gamma(\beta + 1) \sum_{k=0}^{\infty} \binom{-\lambda}{2k} \left[\frac{1}{(2k + \mu)^\beta} + \frac{1}{(2k + \sigma)^\beta} \right]. \end{aligned}$$

In view of (121) and (129), we have

$$\|T_1\| = \|T_2\| = 2\Gamma(\beta + 1) \sum_{k=0}^{\infty} \binom{-\lambda}{2k} \left[\frac{1}{(2k + \mu)^\beta} + \frac{1}{(2k + \sigma)^\beta} \right]. \quad (139)$$

(c) Set

$$H(t) = K_\lambda(1, t) = \frac{(\max\{1, |t|\})^\beta}{|1 + t|^{\lambda+\beta}} \quad (\beta < 1, \mu, \sigma > 0, \mu + \sigma = \lambda < 1 - \beta).$$

Then we have the kernels

$$H(xy) = \frac{(\max\{1, |xy|\})^\beta}{|1 + xy|^{\lambda+\beta}}, \quad K_\lambda(x, y) = \frac{(\max\{|x|, |y|\})^\beta}{|x + y|^{\lambda+\beta}}$$

and obtain the constant factors

$$\begin{aligned} K(\sigma) &= K_\lambda(\sigma) = \int_{-\infty}^{\infty} \frac{(\max\{1, |t|\})^\beta}{|1 + t|^{\lambda+\beta}} |t|^{\sigma-1} dt \\ &= \int_0^{\infty} \frac{(\max\{1, t\})^\beta}{|1 - t|^{\lambda+\beta}} t^{\sigma-1} dt + \int_0^{\infty} \frac{(\max\{1, t\})^\beta}{(1 + t)^{\lambda+\beta}} t^{\sigma-1} dt \\ &= \int_0^1 \left[\frac{1}{(1 - t)^{\lambda+\beta}} + \frac{1}{(1 + t)^{\lambda+\beta}} \right] (t^{\mu-1} + t^{\sigma-1}) dt \\ &= B(1 - \lambda - \beta, \mu) + B(1 - \lambda - \beta, \sigma) + \int_0^1 \frac{t^{\mu-1} + t^{\sigma-1}}{(1 + t)^{\lambda+\beta}} dt \in \mathbf{R}_+. \end{aligned}$$

By Taylor's formula, we still can obtain

$$\begin{aligned}
& \int_0^1 \frac{t^{\mu-1} + t^{\sigma-1}}{(1+t)^{\lambda+\beta}} dt = \int_0^1 \sum_{k=0}^{\infty} \binom{-\lambda-\beta}{k} (t^{k+\mu-1} + t^{k+\sigma-1}) dt \\
&= \int_0^1 \sum_{k=0}^{\infty} (-1)^k \binom{\lambda+\beta+k-1}{k} (t^{k+\mu-1} + t^{k+\sigma-1}) dt \\
&= \int_0^1 \sum_{k=0}^{\infty} \left[\binom{\lambda+\beta+2k-1}{2k} - \binom{\lambda+\beta+2k}{2k+1} t \right] (t^{2k+\mu-1} + t^{2k+\sigma-1}) dt.
\end{aligned}$$

Since we find

$$\begin{aligned}
\binom{\lambda+\beta+2k-1}{2k} - \binom{\lambda+\beta+2k}{2k+1} t &= \binom{\lambda+\beta+2k-1}{2k} - \frac{(\lambda+\beta+2k)t}{2k+1} \binom{\lambda+\beta+2k-1}{2k} \\
&= \left[1 - \frac{(\lambda+\beta+2k)t}{2k+1} \right] \binom{\lambda+\beta+2k-1}{2k},
\end{aligned}$$

there exists a number $k_0 \in \mathbf{N}_0 = \mathbf{N} \cup \{0\}$, such that $\lambda + \beta + 2k_0 > 0$, and for any $s \in \mathbf{N}$,

$$\begin{aligned}
& \binom{\lambda+\beta+2(k_0+s)-1}{2(k_0+s)} - \binom{\lambda+\beta+2(k_0+s)+1}{2(k_0+s)+1} t \\
&= \left[1 - \frac{(\lambda+\beta+2k_0+2s)t}{2(k_0+s)+1} \right] \binom{\lambda+\beta+2(k_0+s)-1}{2(k_0+s)} = \left[1 - \frac{(\lambda+\beta+2k_0+2s)t}{2(k_0+s)+1} \right] \\
&\quad \times \frac{\lambda+\beta+2k_0+2s-1}{2k_0+2s} \dots \frac{\lambda+\beta+2k_0}{2k_0+1} \binom{\lambda+\beta+2k_0-1}{2k_0}.
\end{aligned}$$

For $t \in (0, 1]$, we get

$$\begin{aligned}
1 - \frac{(\lambda+\beta+2k_0+2s)t}{2(k_0+s)+1} &\geq 1 - \frac{\lambda+\beta+2k_0+2s}{2(k_0+s)+1} \\
&= \frac{1-\lambda-\beta}{2(k_0+s)+1} > 0.
\end{aligned}$$

Then it follows that for any $s \in \mathbf{N}$,

$$\operatorname{sgn} \left(\binom{\lambda+\beta+2(k_0+s)-1}{2(k_0+s)} - \binom{\lambda+\beta+2(k_0+s)+1}{2(k_0+s)+1} t \right) = \operatorname{sgn} \left(\binom{\lambda+\beta+2k_0-1}{2k_0} \right).$$

Hence by Lebesgue term by term integration theorem, we have

$$\begin{aligned}
K(\sigma) &= B(1-\lambda-\beta, \mu) + B(1-\lambda-\beta, \sigma) \\
&\quad + \sum_{k=0}^{\infty} \binom{-\lambda-\beta}{k} \int_0^1 (t^{k+\mu-1} + t^{k+\sigma-1}) dt \\
&= B(1-\lambda-\beta, \mu) + B(1-\lambda-\beta, \sigma) \\
&\quad + \sum_{k=0}^{\infty} \binom{-\lambda-\beta}{k} \left(\frac{1}{k+\mu} + \frac{1}{k+\sigma} \right).
\end{aligned}$$

In view of (121) and (129), we have

$$\begin{aligned} \|T_1\| = \|T_2\| &= B(1 - \lambda - \beta, \beta + \mu) + B(1 - \lambda - \beta, \beta + \sigma) \\ &+ \sum_{k=0}^{\infty} \binom{-\lambda - \beta}{k} \left(\frac{1}{k + \mu} + \frac{1}{k + \sigma} \right). \end{aligned} \quad (140)$$

For (a)-(c), we can obtain the equivalent inequalities with the kernels and the best possible constant factors in Theorem 5-8. Setting $\delta_0 = \frac{\sigma}{2} > 0$, we also obtain the equivalent reverse inequalities with the kernels and the best possible constant factors in Theorem 5-8.

(d) Set

$$H(t) = K_\lambda(1, t) = \frac{(\min\{1, |t|\})^\beta}{|1+t|^{\lambda+\beta}},$$

with $\beta > -1, \mu, \sigma > -\beta, \mu + \sigma = \lambda < 1 - \beta$. Then we have the kernels

$$H(xy) = \frac{(\min\{1, |xy|\})^\beta}{|1+xy|^{\lambda+\beta}}, \quad K_\lambda(x, y) = \frac{(\min\{|x|, |y|\})^\beta}{|x+y|^{\lambda+\beta}}$$

and obtain the constant factors

$$\begin{aligned} K(\sigma) = K_\lambda(\sigma) &= \int_{-\infty}^{\infty} \frac{(\min\{1, |t|\})^\beta}{|1+t|^{\lambda+\beta}} |t|^{\sigma-1} dt \\ &= \int_0^{\infty} \frac{(\min\{1, t\})^\beta}{|1-t|^{\lambda+\beta}} t^{\sigma-1} dt + \int_0^{\infty} \frac{(\min\{1, t\})^\beta}{(1+t)^{\lambda+\beta}} t^{\sigma-1} dt \\ &= \int_0^1 \left[\frac{1}{(1-t)^{\lambda+\beta}} + \frac{1}{(1+t)^{\lambda+\beta}} \right] (t^{\beta+\mu-1} + t^{\beta+\sigma-1}) dt \\ &= B(1 - \lambda - \beta, \beta + \mu) + B(1 - \lambda - \beta, \beta + \sigma) \\ &+ \int_0^1 \frac{t^{\beta+\mu-1} + t^{\beta+\sigma-1}}{(1+t)^{\lambda+\beta}} dt \in \mathbf{R}_+. \end{aligned}$$

Similarly to the method followed in (c), we find

$$\begin{aligned} \int_0^1 \frac{t^{\beta+\mu-1} + t^{\beta+\sigma-1}}{(1+t)^{\lambda+\beta}} dt &= \sum_{k=0}^{\infty} \binom{-\lambda - \beta}{k} \int_0^1 (t^{k+\beta+\mu-1} + t^{k+\beta+\sigma-1}) dt \\ &= \sum_{k=0}^{\infty} \binom{-\lambda - \beta}{k} \left(\frac{1}{k + \beta + \mu} + \frac{1}{k + \beta + \sigma} \right). \end{aligned}$$

By the above results, (121) and (129), we have

$$\begin{aligned} \|T_1\| = \|T_2\| &= B(1 - \lambda - \beta, \beta + \mu) + B(1 - \lambda - \beta, \beta + \sigma) \\ &+ \sum_{k=0}^{\infty} \binom{-\lambda - \beta}{k} \left(\frac{1}{k + \beta + \mu} + \frac{1}{k + \beta + \sigma} \right). \end{aligned} \quad (141)$$

Then in (d), we can obtain the equivalent inequalities with the kernels and the best possible constant factors in Theorem 5-8. Setting $\delta_0 = \frac{\sigma+\beta}{2} > 0$, we can still obtain the equivalent reverse inequalities with the kernels and the best possible constant factors in Theorem 5-8.

Example 6. Set

$$H(t) = K_2(1, t) = \frac{1}{1 + 2bt + (ct)^2} \quad (|b| < |c|, \mu = \sigma = 1).$$

Then we have the kernels

$$H(xy) = \frac{1}{1 + 2bxy + (cxy)^2}, \quad K_\lambda(x, y) = \frac{1}{x^2 + 2bxy + (cy)^2}$$

and obtain the constant factors

$$\begin{aligned} K(\sigma) &= K_2(\sigma) = \int_{-\infty}^{\infty} \frac{1}{1 + 2bt + (ct)^2} dt \\ &= \frac{2}{\sqrt{4c^2 - 4b^2}} \arctan \frac{2c^2t + 2b}{\sqrt{4c^2 - 4b^2}} \Big|_{-\infty}^{\infty} = \frac{\pi}{\sqrt{c^2 - b^2}} \in \mathbf{R}_+. \end{aligned}$$

In view of (121) and (129), we have (cf. [13])

$$\|T_1\| = \|T_2\| = \frac{\pi}{\sqrt{c^2 - b^2}}.$$

In particular, for $c = 1, b = \cos \alpha$ ($0 < \alpha < \pi$), we have

$$\|T_1\| = \|T_2\| = \frac{\pi}{\sin \alpha}.$$

Example 7. (a) Set

$$H(t) = K_2(1, t) = \min_{i \in \{1,2\}} \frac{1}{1 + 2t \cos \alpha_i + t^2},$$

with $0 < \alpha_1 \leq \alpha_2 < \pi, 0 < \sigma < 2$. Then we have the kernels

$$H(xy) = \min_{i \in \{1,2\}} \frac{1}{1 + 2xy \cos \alpha_i + (xy)^2}, \quad K_2(x, y) = \min_{i \in \{1,2\}} \frac{1}{x^2 + 2xy \cos \alpha_i + y^2}$$

and obtain the constant factors

$$\begin{aligned} K(\sigma) &= K_2(\sigma) = \int_{-\infty}^{\infty} \min_{i \in \{1,2\}} \frac{1}{1 + 2t \cos \alpha_i + t^2} |t|^{\sigma-1} dt \\ &= \int_0^{\infty} \min_{i \in \{1,2\}} \frac{t^{\sigma-1}}{1 + 2t \cos \alpha_i + t^2} dt + \int_0^{\infty} \min_{i \in \{1,2\}} \frac{t^{\sigma-1}}{1 - 2t \cos \alpha_i + t^2} dt \end{aligned}$$

$$= \int_0^\infty \frac{t^{\sigma-1}}{1+2t \cos \alpha_1 + t^2} dt + \int_0^\infty \frac{t^{\sigma-1}}{1+2t \cos(\pi - \alpha_2) + t^2} dt.$$

Set

$$f(z) = \frac{1}{1+2z \cos \alpha_1 + z^2}.$$

Then

$$z_1 = -e^{i\alpha_1}, \quad z_2 = -e^{-i\alpha_1}$$

are the poles of order 1. Setting

$$\varphi_1(z) = (z - z_1)f(z) = \frac{1}{z - z_2}, \quad \varphi_2(z) = (z - z_2)f(z) = \frac{1}{z - z_1},$$

by (63), we have

$$\begin{aligned} & \int_0^\infty \frac{t^{\sigma-1}}{1+2t \cos \alpha_1 + t^2} dt = \int_0^\infty f(t)t^{\sigma-1} dt \\ &= \frac{\pi}{\sin \pi \sigma} [(-z_1)^{\sigma-1} \varphi_1(z_1) + (-z_2)^{\sigma-1} \varphi_2(z_2)] \\ &= \frac{\pi}{\sin \pi \sigma} \left[\frac{e^{i\alpha_1(\sigma-1)}}{-e^{i\alpha_1} + e^{-i\alpha_1}} + \frac{e^{-i\alpha_1(\sigma-1)}}{-e^{-i\alpha_1} + e^{i\alpha_1}} \right] = \frac{\pi \sin \alpha_1 (1 - \sigma)}{\sin \pi \sigma \sin \alpha_1}. \end{aligned}$$

Similarly, it follows that

$$\begin{aligned} & \int_0^\infty \frac{t^{\sigma-1} dt}{1+2t \cos(\pi - \alpha_2) + t^2} \\ &= \frac{\pi \sin(\pi - \alpha_2)(1 - \sigma)}{\sin \pi \sigma \sin(\pi - \alpha_2)} = \frac{\pi \sin(\pi - \alpha_2)(1 - \sigma)}{\sin \pi \sigma \sin \alpha_2}, \end{aligned}$$

and then

$$K(\sigma) = \frac{\pi}{\sin \pi \sigma} \left[\frac{\sin \alpha_1 (1 - \sigma)}{\sin \alpha_1} + \frac{\sin(\pi - \alpha_2)(1 - \sigma)}{\sin \alpha_2} \right] \in \mathbf{R}_+.$$

In view of (121) and (129), we have (cf. [21])

$$\|T_1\| = \|T_2\| = \frac{\pi}{\sin \pi \sigma} \left[\frac{\sin \alpha_1 (1 - \sigma)}{\sin \alpha_1} + \frac{\sin(\pi - \alpha_2)(1 - \sigma)}{\sin \alpha_2} \right]. \quad (142)$$

Then in (a), we can obtain the equivalent inequalities with the kernels and the best possible constant factors in Theorem 5-8. Setting $\delta_0 = \frac{\sigma}{2} > 0$, we can still obtain the equivalent reverse inequalities with the kernels and the best possible constant factors in Theorem 5-8.

In particular, if $\alpha_1 = \alpha_2 = \alpha \in (0, \pi)$, then

$$H(t) = K_2(1, t) = \frac{1}{1+2t \cos \alpha + t^2}$$

and

$$\|T_1\| = \|T_2\| = \frac{\pi}{\sin \pi \sigma \sin \alpha} [\sin \alpha (1 - \sigma) + \sin(\pi - \alpha)(1 - \sigma)].$$

(b) Set

$$H(t) = K_0(1, t) = \min_{i \in \{1, 2\}} \frac{\min\{1, |t|\}}{\sqrt{1 + 2t \cos \alpha_i + t^2}},$$

with $0 < \alpha_1 \leq \alpha_2 < \pi$, $\sigma = \mu = 0$. Then we have the kernels

$$H(xy) = \min_{i \in \{1, 2\}} \frac{\min\{1, |xy|\}}{\sqrt{1 + 2xy \cos \alpha_i + (xy)^2}},$$

$$K_0(x, y) = \min_{i \in \{1, 2\}} \frac{\min\{|x|, |y|\}}{\sqrt{x^2 + 2xy \cos \alpha_i + y^2}}$$

and obtain the constant factors

$$\begin{aligned} K(0) &= K_0(0) = \int_{-\infty}^{\infty} \min_{i \in \{1, 2\}} \frac{\min\{1, |t|\}}{\sqrt{1 + 2t \cos \alpha_i + t^2}} |t|^{-1} dt \\ &= \int_0^{\infty} \min_{i \in \{1, 2\}} \frac{\min\{1, t\} t^{-1}}{\sqrt{1 + 2t \cos \alpha_i + t^2}} dt \\ &\quad + \int_0^{\infty} \min_{i \in \{1, 2\}} \frac{\min\{1, t\} t^{-1}}{\sqrt{1 - 2t \cos \alpha_i + t^2}} dt \\ &= \int_0^{\infty} \frac{\min\{1, t\} t^{-1} dt}{\sqrt{1 + 2t \cos \alpha_1 + t^2}} + \int_0^{\infty} \frac{\min\{1, t\} t^{-1} dt}{\sqrt{1 + 2t \cos(\pi - \alpha_2) + t^2}} \\ &= 2 \left[\int_0^1 \frac{dt}{\sqrt{1 + 2t \cos \alpha_1 + t^2}} + \int_0^1 \frac{dt}{\sqrt{1 + 2t \cos(\pi - \alpha_2) + t^2}} \right]. \end{aligned}$$

We get

$$\begin{aligned} &\int_0^1 \frac{dt}{\sqrt{1 + 2t \cos \alpha_1 + t^2}} \\ &= \ln(2t + 2 \cos \alpha_1 + 2\sqrt{1 + 2t \cos \alpha_1 + t^2}) \Big|_0^1 = \ln \left(1 + \sec \frac{\alpha_1}{2} \right), \end{aligned}$$

and by the same way,

$$\int_0^1 \frac{dt}{\sqrt{1 + 2t \cos(\pi - \alpha_2) + t^2}} = \ln \left(1 + \sec \frac{\pi - \alpha_2}{2} \right) = \ln \left(1 + \csc \frac{\alpha_2}{2} \right).$$

Then it follows that

$$K(0) = K_0(0) = 2 \ln \left(1 + \sec \frac{\alpha_1}{2} \right) \left(1 + \csc \frac{\alpha_2}{2} \right).$$

By (121) and (129), we have (cf. [20])

$$\|T_1\| = \|T_2\| = 2\ln\left(1 + \sec\frac{\alpha_1}{2}\right)\left(1 + \csc\frac{\alpha_2}{2}\right). \quad (143)$$

In particular, if $\alpha_1 = \alpha_2 = \alpha \in (0, \pi)$, then

$$H(t) = K_0(1, t) = \frac{\min\{1, |t|\}}{\sqrt{1 + 2t \cos \alpha + t^2}}$$

and

$$\|T_1\| = \|T_2\| = 2\ln\left(1 + \sec\frac{\alpha}{2}\right)\left(1 + \csc\frac{\alpha}{2}\right).$$

Example 8. Set

$$H(t) = K_0(1, t) = \left| \ln \frac{1 + 2t \cos \alpha_1 + t^2}{1 + 2t \cos \alpha_2 + t^2} \right|,$$

with $0 < \alpha_1 \leq \alpha_2 < \pi$, $\mu = -\sigma \in (0, 1)$. Then we have the kernels

$$H(xy) = \left| \ln \frac{1 + 2xy \cos \alpha_1 + (xy)^2}{1 + 2xy \cos \alpha_2 + (xy)^2} \right|,$$

$$K_0(x, y) = \left| \ln \frac{x^2 + 2xy \cos \alpha_1 + y^2}{x^2 + 2xy \cos \alpha_2 + y^2} \right|$$

and obtain the constant factors

$$\begin{aligned} K(\sigma) &= K_0(\sigma) = \int_{-\infty}^{\infty} \left| \ln \frac{1 + 2t \cos \alpha_1 + t^2}{1 + 2t \cos \alpha_2 + t^2} \right| \cdot |t|^{\sigma-1} dt \\ &= \int_0^{\infty} t^{\sigma-1} \ln \frac{1 + 2t \cos \alpha_1 + t^2}{1 + 2t \cos \alpha_2 + t^2} dt \\ &\quad + \int_0^{\infty} t^{\sigma-1} \ln \frac{1 - 2t \cos \alpha_2 + t^2}{1 - 2t \cos \alpha_1 + t^2} dt \\ &= \int_0^{\infty} t^{\sigma-1} \ln \frac{1 + 2t \cos \alpha_1 + t^2}{1 + 2t \cos \alpha_2 + t^2} dt \\ &\quad + \int_0^{\infty} t^{\sigma-1} \ln \frac{1 + 2t \cos(\pi - \alpha_2) + t^2}{1 + 2t \cos(\pi - \alpha_1) + t^2} dt. \end{aligned}$$

We find

$$\begin{aligned} I_1 &:= \int_0^{\infty} t^{\sigma-1} \ln \frac{1 + 2t \cos \alpha_1 + t^2}{1 + 2t \cos \alpha_2 + t^2} dt = \frac{1}{\sigma} \int_0^{\infty} \ln \frac{1 + 2t \cos \alpha_1 + t^2}{1 + 2t \cos \alpha_2 + t^2} dt^\sigma \\ &= \frac{1}{\sigma} \left[t^\sigma \ln \frac{1 + 2t \cos \alpha_1 + t^2}{1 + 2t \cos \alpha_2 + t^2} \Big|_0^{\infty} \right. \\ &\quad \left. - 2 \int_0^{\infty} t^\sigma \left(\frac{\cos \alpha_1 + t}{1 + 2t \cos \alpha_1 + t^2} - \frac{\cos \alpha_2 + t}{1 + 2t \cos \alpha_2 + t^2} \right) dt \right] \\ &= \frac{-2}{\sigma} \left[\int_0^{\infty} \frac{(\cos \alpha_1 + t)t^\sigma}{1 + 2t \cos \alpha_1 + t^2} dt - \int_0^{\infty} \frac{(\cos \alpha_2 + t)t^\sigma}{1 + 2t \cos \alpha_2 + t^2} dt \right] \end{aligned}$$

$$= \frac{-2}{\sigma} (I_1^{(1)} - I_1^{(2)}),$$

$$I_1^{(i)} = \int_0^\infty \frac{(\cos \alpha_i + t)t^{(\sigma+1)-1}}{1 + 2t \cos \alpha_i + t^2} dt \quad (i = 1, 2).$$

By (63), we have

$$I_1^{(i)} = \frac{\pi}{\sin \pi(\sigma + 1)} \left(e^{i\alpha_i \sigma} \frac{\cos \alpha_i - e^{i\alpha_i}}{-e^{i\alpha_i} + e^{-i\alpha_i}} + e^{-i\alpha_i \sigma} \frac{\cos \alpha_i - e^{-i\alpha_i}}{-e^{-i\alpha_i} + e^{i\alpha_i}} \right)$$

$$= \frac{\pi \cos \alpha_i \sigma}{\sin \pi(\sigma + 1)} \quad (i = 1, 2),$$

and then

$$I_1 = \frac{-2}{\sigma} \left[\frac{\pi \cos \alpha_1 \sigma}{\sin \pi(\sigma + 1)} - \frac{\pi \cos \alpha_2 \sigma}{\sin \pi(\sigma + 1)} \right]$$

$$= \frac{-2\pi(\cos \alpha_1 \sigma - \cos \alpha_2 \sigma)}{\sigma \sin \pi(\sigma + 1)}$$

$$= \frac{4\pi}{\sigma \sin \pi(\sigma + 1)} \sin \frac{\alpha_1 + \alpha_2}{2} \sigma \sin \frac{\alpha_1 - \alpha_2}{2} \sigma.$$

Similarly, we have

$$I_2 := \int_0^\infty t^{\sigma-1} \ln \frac{1 + 2t \cos(\pi - \alpha_2) + t^2}{1 + 2t \cos(\pi - \alpha_1) + t^2} dt$$

$$= \frac{4\pi}{\sigma \sin \pi(\sigma + 1)} \sin \left(\pi - \frac{\alpha_1 + \alpha_2}{2} \right) \sigma \sin \frac{\alpha_1 - \alpha_2}{2} \sigma,$$

and then

$$K(\sigma) = K_0(\sigma) = \frac{4\pi}{\sigma \sin \pi(\sigma + 1)} \sin \frac{\alpha_1 + \alpha_2}{2} \sigma$$

$$\times \left[\sin \frac{\alpha_1 - \alpha_2}{2} \sigma + \sin \left(\pi - \frac{\alpha_1 + \alpha_2}{2} \right) \sigma \right]$$

$$= \frac{-4\pi \sin \frac{\sigma}{2} (\alpha_1 - \alpha_2)}{\sigma \cos \pi(\sigma/2)} \cos \frac{\sigma}{2} (\alpha_1 + \alpha_2 - \pi) \in \mathbf{R}_+.$$

In view of (121) and (129), we have (cf. [18])

$$\|T_1\| = \|T_2\| = \frac{-4\pi \sin \frac{\sigma}{2} (\alpha_1 - \alpha_2)}{\sigma \cos \pi(\sigma/2)} \cos \frac{\sigma}{2} (\alpha_1 + \alpha_2 - \pi). \quad (144)$$

Then we can obtain the equivalent inequalities with the kernels and the best possible constant factors in Theorem 5-8. Setting $\delta_0 = \frac{-\sigma}{2} > 0$, we still can obtain the equivalent reverse inequalities with the kernels and the best possible constant factors in Theorem 5-8.

Remark 9. Since $K(0^-) = 2\pi(\alpha_2 - \alpha_1) \in \mathbf{R}_+$, then (144) is valid for $\sigma \in (-1, 0]$.

Example 9. (a) For

$$\gamma \in \{a; a = \frac{1}{2k-1}, 2k+1 (k \in \mathbf{N})\},$$

we set

$$H(t) = K_\lambda(1, t) = \min_{i \in \{1, 2\}} \frac{1}{(1 + t^\gamma \cos \alpha_i + |t|^\gamma)^{\lambda/\gamma}},$$

where $0 < \alpha_1 \leq \alpha_2 < \pi$, $\mu, \sigma > 0$, $\mu + \sigma = \lambda$. Then we have the kernels

$$H(xy) = \min_{i \in \{1, 2\}} \frac{1}{(1 + (xy)^\gamma \cos \alpha_i + |xy|^\gamma)^{\lambda/\gamma}},$$

$$K_\lambda(x, y) = \min_{i \in \{1, 2\}} \frac{1}{(|x|^\gamma + y^\gamma \operatorname{sgn}(x) \cos \alpha_i + |y|^\gamma)^{\lambda/\gamma}}$$

and obtain the constant factors

$$\begin{aligned} K(\sigma) &= K(\sigma) = K_\lambda(\sigma) = \int_{-\infty}^{\infty} \min_{i \in \{1, 2\}} \frac{1}{(1 + t^\gamma \cos \alpha_i + |t|^\gamma)^{\lambda/\gamma}} |t|^{\sigma-1} dt \\ &= \int_0^{\infty} \frac{t^{\sigma-1} dt}{[1 + t^\gamma(1 + \cos \alpha_1)]^{\lambda/\gamma}} + \int_0^{\infty} \frac{t^{\sigma-1} dt}{[1 + t^\gamma(1 - \cos \alpha_2)]^{\lambda/\gamma}} \\ &= \frac{1}{\gamma} \left[\frac{1}{(1 + \cos \alpha_1)^{\lambda/\gamma}} + \frac{1}{(1 - \cos \alpha_2)^{\lambda/\gamma}} \right] \int_0^{\infty} \frac{u^{(\sigma/\gamma)-1} du}{(1+u)^{\lambda/\gamma}} \\ &= \frac{1}{\gamma 2^{\sigma/\gamma}} \left[\left(\sec \frac{\alpha_1}{2}\right)^{\frac{2\sigma}{\gamma}} + \left(\csc \frac{\alpha_2}{2}\right)^{\frac{2\sigma}{\gamma}} \right] B\left(\frac{\mu}{\gamma}, \frac{\sigma}{\gamma}\right). \end{aligned}$$

In view of (121) and (129), we have

$$\|T_1\| = \|T_2\| = \frac{1}{\gamma 2^{\sigma/\gamma}} \left[\left(\sec \frac{\alpha_1}{2}\right)^{\frac{2\sigma}{\gamma}} + \left(\csc \frac{\alpha_2}{2}\right)^{\frac{2\sigma}{\gamma}} \right] B\left(\frac{\mu}{\gamma}, \frac{\sigma}{\gamma}\right). \quad (145)$$

In particular, if $\alpha_1 = \alpha_2 = \alpha \in (0, \pi)$, then it follows that

$$H(t) = K_\lambda(1, t) = \frac{1}{(1 + t^\gamma \cos \alpha + |t|^\gamma)^{\lambda/\gamma}},$$

and

$$\|T_1\| = \|T_2\| = \frac{1}{\gamma 2^{\sigma/\gamma}} \left[\left(\sec \frac{\alpha}{2}\right)^{\frac{2\sigma}{\gamma}} + \left(\csc \frac{\alpha}{2}\right)^{\frac{2\sigma}{\gamma}} \right] B\left(\frac{\mu}{\gamma}, \frac{\sigma}{\gamma}\right).$$

(b) For

$$\gamma \in \{a; a = \frac{1}{2k-1}, 2k+1 (k \in \mathbf{N})\},$$

we set

$$H(t) = K_\lambda(1, t) = \min_{i \in \{1, 2\}} \frac{1}{|1 - t^\gamma \cos \alpha_i - |t|^\gamma|^{\lambda/\gamma}},$$

where $0 < \alpha_1 \leq \alpha_2 < \pi$, $\mu, \sigma > 0$, $\mu + \sigma = \lambda < \gamma$. Then we have the kernels

$$H(xy) = \min_{i \in \{1, 2\}} \frac{1}{|1 - (xy)^\gamma \cos \alpha_i - |xy|^\gamma|^{\lambda/\gamma}},$$

$$K_\lambda(x, y) = \min_{i \in \{1, 2\}} \frac{1}{||x|^\gamma - y^\gamma \operatorname{sgn}(x) \cos \alpha_i - |y|^\gamma|^{\lambda/\gamma}}$$

and obtain the constant factors

$$\begin{aligned} K(\sigma) &= K_\lambda(\sigma) = \int_{-\infty}^{\infty} \min_{i \in \{1, 2\}} \frac{1}{|1 - t^\gamma \cos \alpha_i - |t|^\gamma|^{\lambda/\gamma}} |t|^{\sigma-1} dt \\ &= \int_0^{\infty} \min_{i \in \{1, 2\}} \frac{1}{|1 - t^\gamma(1 + \cos \alpha_i)|^{\lambda/\gamma}} t^{\sigma-1} dt \\ &\quad + \int_0^{\infty} \min_{i \in \{1, 2\}} \frac{1}{|1 - t^\gamma(1 - \cos \alpha_i)|^{\lambda/\gamma}} t^{\sigma-1} dt \\ &= \frac{1}{\gamma} \left[\int_0^{\infty} \frac{1}{(1 + \cos \alpha_1)^{\lambda/\gamma}} \int_0^{\infty} \frac{u^{(\sigma/\gamma)-1} du}{|1 - u|^{\lambda/\gamma}} \right] \\ &\quad + \frac{1}{\gamma} \left[\int_0^{\infty} \frac{1}{(1 - \cos \alpha_2)^{\lambda/\gamma}} \int_0^{\infty} \frac{u^{(\sigma/\gamma)-1} du}{|1 - u|^{\lambda/\gamma}} \right] \\ &= \frac{1}{\gamma 2^{\sigma/\gamma}} \left[\left(\sec \frac{\alpha_1}{2} \right)^{\frac{2\sigma}{\gamma}} + \left(\csc \frac{\alpha_2}{2} \right)^{\frac{2\sigma}{\gamma}} \right] \\ &\quad \times \left[B\left(1 - \frac{\lambda}{\gamma}, \frac{\mu}{\gamma}\right) + B\left(1 - \frac{\lambda}{\gamma}, \frac{\sigma}{\gamma}\right) \right]. \end{aligned}$$

In view of (121) and (129), we have

$$\begin{aligned} \|T_1\| = \|T_2\| &= \frac{1}{\gamma 2^{\sigma/\gamma}} \left[\left(\sec \frac{\alpha_1}{2} \right)^{\frac{2\sigma}{\gamma}} + \left(\csc \frac{\alpha_2}{2} \right)^{\frac{2\sigma}{\gamma}} \right] \\ &\quad \times \left[B\left(1 - \frac{\lambda}{\gamma}, \frac{\mu}{\gamma}\right) + B\left(1 - \frac{\lambda}{\gamma}, \frac{\sigma}{\gamma}\right) \right]. \end{aligned} \tag{146}$$

In particular, if $\alpha_1 = \alpha_2 = \alpha \in (0, \pi)$, then we have

$$H(t) = K_\lambda(1, t) = \frac{1}{|1 - t^\gamma \cos \alpha - |t|^\gamma|^{\lambda/\gamma}},$$

and

$$\|T_1\| = \|T_2\| = \frac{1}{\gamma 2^{\sigma/\gamma}} \left[\left(\sec \frac{\alpha}{2} \right)^{\frac{2\sigma}{\gamma}} + \left(\csc \frac{\alpha}{2} \right)^{\frac{2\sigma}{\gamma}} \right]$$

$$\times \left[B\left(1 - \frac{\lambda}{\gamma}, \frac{\mu}{\gamma}\right) + B\left(1 - \frac{\lambda}{\gamma}, \frac{\sigma}{\gamma}\right) \right].$$

(c) For

$$\gamma \in \{a; a = \frac{1}{2k-1}, 2k+1 \ (k \in \mathbf{N})\},$$

we set

$$H(t) = K_\lambda(1, t) = \min_{i \in \{1, 2\}} \frac{\ln(t^\gamma \cos \alpha_i + |t|^\gamma)}{(t^\gamma \cos \alpha_i + |t|^\gamma)^{\lambda/\gamma} - 1},$$

with $0 < \alpha_1 \leq \alpha_2 < \pi$, $\mu, \sigma > 0$, $\mu + \sigma = \lambda$. Then we have the kernels

$$H(xy) = \min_{i \in \{1, 2\}} \frac{\ln((xy)^\gamma \cos \alpha_i + |xy|^\gamma)}{((xy)^\gamma \cos \alpha_i + |xy|^\gamma)^{\lambda/\gamma} - 1},$$

$$K_\lambda(x, y) = \min_{i \in \{1, 2\}} \frac{\ln\left(\left(\frac{x}{y}\right)^\gamma \cos \alpha_i + \left|\frac{x}{y}\right|^\gamma\right)}{(y^\gamma \operatorname{sgn}(x) \cos \alpha_i + |y|^\gamma)^{\lambda/\gamma} - |x|^\lambda}$$

and obtain the constant factors

$$\begin{aligned} K(\sigma) &= K_\lambda(\sigma) = \int_{-\infty}^{\infty} \min_{i \in \{1, 2\}} \frac{\ln(t^\gamma \cos \alpha_i + |t|^\gamma)}{(t^\gamma \cos \alpha_i + |t|^\gamma)^{\lambda/\gamma} - 1} |t|^{\sigma-1} dt \\ &= \int_0^{\infty} \min_{i \in \{1, 2\}} \frac{\ln[t^\gamma(1 + \cos \alpha_i)]}{[t^\gamma(1 + \cos \alpha_i)]^{\lambda/\gamma} - 1} t^{\sigma-1} dt \\ &\quad + \int_0^{\infty} \min_{i \in \{1, 2\}} \frac{\ln[t^\gamma(1 - \cos \alpha_i)]}{[t^\gamma(1 - \cos \alpha_i)]^{\lambda/\gamma} - 1} t^{\sigma-1} dt \\ &= \frac{\gamma}{\lambda^2} \left[\int_0^{\infty} \frac{1}{(1 + \cos \alpha_1)^{\sigma/\gamma}} \int_0^{\infty} \frac{\ln u}{u-1} u^{\frac{\sigma}{\lambda}-1} du \right] \\ &\quad + \frac{\gamma}{\lambda^2} \left[\int_0^{\infty} \frac{1}{(1 - \cos \alpha_2)^{\sigma/\gamma}} \int_0^{\infty} \frac{\ln u}{u-1} u^{\frac{\sigma}{\lambda}-1} du \right] \\ &= \frac{\gamma}{\lambda^2 2^{\sigma/\gamma}} \left[\left(\sec \frac{\alpha_1}{2} \right)^{\frac{2\sigma}{\gamma}} + \left(\csc \frac{\alpha_2}{2} \right)^{\frac{2\sigma}{\gamma}} \right] \left[\frac{\pi}{\sin \pi(\sigma/\lambda)} \right]^2. \end{aligned}$$

By (121) and (129), we have

$$\|T_1\| = \|T_2\| = \frac{\gamma}{\lambda^2 2^{\sigma/\gamma}} \left[\left(\sec \frac{\alpha_1}{2} \right)^{\frac{2\sigma}{\gamma}} + \left(\csc \frac{\alpha_2}{2} \right)^{\frac{2\sigma}{\gamma}} \right] \left[\frac{\pi}{\sin \pi(\sigma/\lambda)} \right]^2.$$

In particular, if $\alpha_1 = \alpha_2 = \alpha \in (0, \pi)$, then we have

$$H(t) = K_\lambda(1, t) = \frac{\ln(t^\gamma \cos \alpha + |t|^\gamma)}{(t^\gamma \cos \alpha + |t|^\gamma)^{\lambda/\gamma} - 1},$$

and

$$\|T_1\| = \|T_2\| = \frac{\gamma}{\lambda^2 2^{\sigma/\gamma}} \left[\left(\sec \frac{\alpha}{2} \right)^{\frac{2\sigma}{\gamma}} + \left(\csc \frac{\alpha}{2} \right)^{\frac{2\sigma}{\gamma}} \right] \left[\frac{\pi}{\sin \pi (\sigma/\lambda)} \right]^2.$$

(d) For

$$\gamma \in \left\{ a; a = \frac{1}{2k-1}, 2k+1 (k \in \mathbf{N}) \right\},$$

we set

$$H(t) = K_\lambda(1, t) = \min_{i \in \{1,2\}} \frac{1}{\max\{1, (t^\gamma \cos \alpha_i + |t|^\gamma)^{\lambda/\gamma}\}},$$

where $0 < \alpha_1 \leq \alpha_2 < \pi$, $\mu, \sigma > 0$, $\mu + \sigma = \lambda$. Then we have the kernels

$$H(xy) = \min_{i \in \{1,2\}} \frac{1}{\max\{1, [(xy)^\gamma \cos \alpha_i + |xy|^\gamma]^{\lambda/\gamma}\}},$$

$$K_\lambda(x, y) = \min_{i \in \{1,2\}} \frac{1}{\max\{|x|, (y^\gamma \operatorname{sgn}(x) \cos \alpha_i + |y|^\gamma)^{\lambda/\gamma}\}}$$

and obtain the constant factors

$$\begin{aligned} K(\sigma) &= K_\lambda(\sigma) = \int_{-\infty}^{\infty} \min_{i \in \{1,2\}} \frac{1}{\max\{1, (t^\gamma \cos \alpha_i + |t|^\gamma)^{\lambda/\gamma}\}} |t|^{\sigma-1} dt \\ &= \int_0^{\infty} \min_{i \in \{1,2\}} \frac{1}{\max\{1, [t^\gamma(1 + \cos \alpha_i)]^{\lambda/\gamma}\}} t^{\sigma-1} dt \\ &\quad + \int_0^{\infty} \min_{i \in \{1,2\}} \frac{1}{\max\{1, [t^\gamma(1 - \cos \alpha_i)]^{\lambda/\gamma}\}} t^{\sigma-1} dt \\ &= \frac{1}{\lambda} \left[\int_0^{\infty} \min_{i \in \{1,2\}} \frac{1}{(1 + \cos \alpha_i)^{\sigma/\gamma}} \frac{1}{\max\{1, u\}} u^{\frac{\sigma}{\lambda}-1} du \right] \\ &\quad + \frac{1}{\lambda} \left[\int_0^{\infty} \min_{i \in \{1,2\}} \frac{1}{(1 - \cos \alpha_i)^{\sigma/\gamma}} \frac{1}{\max\{1, u\}} u^{\frac{\sigma}{\lambda}-1} du \right] \\ &= \frac{1}{\lambda} \left[\int_0^{\infty} \frac{1}{(1 + \cos \alpha_1)^{\sigma/\gamma}} \int_0^{\infty} \frac{1}{\max\{1, u\}} u^{\frac{\sigma}{\lambda}-1} du \right] \\ &\quad + \frac{1}{\lambda} \left[\int_0^{\infty} \frac{1}{(1 - \cos \alpha_2)^{\sigma/\gamma}} \int_0^{\infty} \frac{1}{\max\{1, u\}} u^{\frac{\sigma}{\lambda}-1} du \right] \\ &= \frac{1}{2^{\sigma/\gamma}} \left[\left(\sec \frac{\alpha_1}{2} \right)^{\frac{2\sigma}{\gamma}} + \left(\csc \frac{\alpha_2}{2} \right)^{\frac{2\sigma}{\gamma}} \right] \frac{\lambda}{\mu \sigma}. \end{aligned}$$

By (121) and (129), we have

$$\|T_1\| = \|T_2\| = \frac{1}{2^{\sigma/\gamma}} \left[\left(\sec \frac{\alpha_1}{2} \right)^{\frac{2\sigma}{\gamma}} + \left(\csc \frac{\alpha_2}{2} \right)^{\frac{2\sigma}{\gamma}} \right] \frac{\lambda}{\mu \sigma}. \quad (147)$$

In particular, if $\alpha_1 = \alpha_2 = \alpha \in (0, \pi)$, then we have

$$H(t) = K_\lambda(1, t) = \frac{1}{\max\{1, (t^\gamma \cos \alpha + |t|^\gamma)^{\lambda/\gamma}\}},$$

and

$$\|T_1\| = \|T_2\| = \frac{1}{2^{\sigma/\gamma}} \left[\left(\sec \frac{\alpha}{2} \right)^{\frac{2\sigma}{\gamma}} + \left(\csc \frac{\alpha}{2} \right)^{\frac{2\sigma}{\gamma}} \right] \frac{\lambda}{\mu\sigma}.$$

Then, for (a)-(d) we can obtain the equivalent inequalities with the kernels and the best possible constant factors in Theorem 5-8. Setting $\delta_0 = \frac{\sigma}{2} > 0$, we can still obtain the equivalent reverse inequalities with the kernels and the best possible constant factors in Theorem 5-8.

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