ON AN EQUIVALENT PROPERTY OF A REVERSE HILBERT-TYPE INTEGRAL INEQUALITY RELATED TO THE EXTENDED HURWITZ-ZETA FUNCTION

MICHAEL TH. RASSIAS[1*] and BICHENG YANG[2]

Abstract. We study some equivalent conditions of a reverse Hilbert-type integral inequality with a particular non-homogeneous kernel and a best possible constant factor related to the extended Hurwitz-zeta function. Some equivalent conditions of a reverse Hilbert-type integral inequality with the particular homogeneous kernel are deduced. We also consider some particular cases.

1. Introduction

In 1925, Hardy [3] proved the following extension of Hilbert's integral inequality (cf. [4]):
For $p > 1, \frac{1}{p} + \frac{1}{q} = 1, f(x), g(y) \geq 0$,

$$0 < \int_0^\infty f^p(x)dx < \infty \ \ and \ \ 0 < \int_0^\infty g^q(y)dy < \infty,$$

the following Hardy-Hilbert inequality holds true:

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} dxdy < \frac{\pi}{\sin(\pi/p)} \left(\int_0^\infty f^p(x)dx \right)^{\frac{1}{p}} \left(\int_0^\infty g^q(y)dy \right)^{\frac{1}{q}}, \tag{1.1}$$

with the best possible constant factor $\frac{\pi}{\sin(\pi/p)}$.
For $p = q = 2$, the inequality (1.1) reduces to Hilbert's integral inequality, which is important in mathematical analysis and its applications (cf. [5], [6]).

In 1934, Hardy et al. extended the inequality (1.1) as follows:
If $p > 1, \frac{1}{p} + \frac{1}{q} = 1, k_1(x, y)$ is a nonnegative homogeneous function of degree -1, such that

$$k_p = \int_0^\infty k_1(u, 1)u^{\frac{-1}{p}} du \in \mathbf{R}_+ = (0, \infty), \tag{1.2}$$

Date: July 28, 2018

*Corresponding author.

2010 *Mathematics Subject Classification.* Primary 26D15; Secondary 47A07, 65B10.

Key words and phrases. reverse Hilbert-type integral inequality; weight function; equivalent form; kernel; Hurwitz zeta function.

then we have the following Hardy-Hilbert-type integral inequality with the best possible constant k_p:

$$\int_0^\infty \int_0^\infty k_1(x,y)f(x)g(y)dxdy$$
$$< \quad k_p \left(\int_0^\infty f^p(x)dx\right)^{\frac{1}{p}} \left(\int_0^\infty g^q(y)dy\right)^{\frac{1}{q}}; \tag{1.3}$$

for $0 < p < 1, \frac{1}{p} + \frac{1}{q} = 1$, the reverse of (1.3) follows (cf. [5], Theorem 319, Theorem 336). A Hilbert-type integral inequality with the non-homogeneous kernel was proved:
If $0 < p < 1, \frac{1}{p} + \frac{1}{q} = 1, h(u) > 0$, $\phi(\sigma) = \int_0^\infty h(u)u^{\sigma-1}du \in \mathbf{R}_+$, then

$$\int_0^\infty \int_0^\infty h(xy)f(x)g(y)dxdy$$
$$< \quad \phi(\frac{1}{p}) \left(\int_0^\infty x^{p-2}f^p(x)dx\right)^{\frac{1}{p}} \left(\int_0^\infty g^q(y)dy\right)^{\frac{1}{q}}, \tag{1.4}$$

with the best possible constant factor $\phi(\frac{1}{p})$ (cf. [5], Theorem 350).

In 1998, by introducing an independent parameter $\lambda > 0$, Yang proved an extension of Hilbert's integral inequality with the kernel $\frac{1}{(x+y)^\lambda}$ (cf. [7], [8]). In 2004, by introducing another pair of conjugate exponents (r,s), Yang [9] proved the following extension of inequality (1.1):
If $\lambda > 0, p, r > 1$, $\frac{1}{p} + \frac{1}{q} = \frac{1}{r} + \frac{1}{s} = 1$, $f(x), g(y) \geq 0$, satisfying

$$0 < \int_0^\infty x^{p(1-\frac{\lambda}{r})-1}f^p(x)dx < \infty$$

and

$$0 < \int_0^\infty y^{q(1-\frac{\lambda}{s})-1}g^q(y)dy < \infty,$$

then

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x^\lambda + y^\lambda}dxdy$$
$$< \quad \frac{\pi}{\lambda\sin(\pi/r)} \left[\int_0^\infty x^{p(1-\frac{\lambda}{r})-1}f^p(x)dx\right]^{\frac{1}{p}} \left[\int_0^\infty y^{q(1-\frac{\lambda}{s})-1}g^q(y)dy\right]^{\frac{1}{q}}, \tag{1.5}$$

with the best possible constant factor $\frac{\pi}{\lambda\sin(\pi/r)}$. In 2005, an extension of (1.1) with the kernel $\frac{1}{(x+y)^\lambda}$ and two pairs of conjugate exponents was proved in [10]. Krnić et al. [12]-[18] provided some extensions and particular cases of (1.1), (1.3) and (1.5) with multi-parameters.

In 2009, Yang showed the following extension of (1.3) and (1.5) (cf. [19], [21]):
If $\lambda_1 + \lambda_2 = \lambda \in \mathbf{R} = (-\infty, \infty)$, $k_\lambda(x,y)$ is a non-negative homogeneous function of degree $-\lambda$, satisfying

$$k_\lambda(ux, uy) = u^{-\lambda}k_\lambda(x,y)(u,x,y > 0),$$

and

$$k(\lambda_1) = \int_0^\infty k_\lambda(u,1)u^{\lambda_1-1}du \in \mathbf{R}_+,$$

then for $p > 1, \frac{1}{p} + \frac{1}{q} = 1$, we have

$$\begin{aligned} &\int_0^\infty \int_0^\infty k_\lambda(x,y)f(x)g(y)dxdy \\ < \;& k(\lambda_1)\left[\int_0^\infty x^{p(1-\lambda_1)-1}f^p(x)dx\right]^{\frac{1}{p}}\left[\int_0^\infty y^{q(1-\lambda_2)-1}g^q(y)dy\right]^{\frac{1}{q}}, \end{aligned} \tag{1.6}$$

with the best possible constant factor $k(\lambda_1)$.
For $0 < p < 1, \frac{1}{p} + \frac{1}{q} = 1$, the reverse of (1.6) follows. . Additionally, the following extension of (1.4) was proved:
For $p > 1, \frac{1}{p} + \frac{1}{q} = 1$, we have

$$\begin{aligned} &\int_0^\infty \int_0^\infty h(xy)f(x)g(y)dxdy \\ < \;& \phi(\sigma)\left(\int_0^\infty x^{p(1-\sigma)-1}f^p(x)dx\right)^{\frac{1}{p}}\left(\int_0^\infty y^{q(1-\sigma)-1}g^q(y)dy\right)^{\frac{1}{q}}, \end{aligned} \tag{1.7}$$

with the best possible constant factor $\phi(\sigma)$.
For $0 < p < 1, \frac{1}{p} + \frac{1}{q} = 1$, the reverse of (1.7) follows (cf. [20]).

Some equivalent inequalities of (1.6) are obtained in [21] . In 2013, Yang [20] studied as well the equivalency of (1.6) and (1.7). In 2017, in [22] and [23] some equivalent condition between a Hilbert-type integral inequality and the related parameters were investigated. For other closely related results the reader is also referred to [1], [2], [11], [23].

In this paper, by the use of techniques of real analysis and weight functions, we consider some equivalent conditions of a reverse of (1.7) in the particular kernel $H(xy) = e^{-\alpha xy}\operatorname{csch}(xy)$ where $0 < p < 1$, with the best possible constant factor related to the extended Hurwitz-zeta function. Some equivalent conditions of the reverse of (1.6) for the particular kernel

$$k_0(x,y) = e^{-\alpha x/y}\csc h(\frac{x}{y})$$

are deduced. We also consider some particular cases as corollaries.

2. An example and two lemmas

Example 2.1. Setting

$$H(u) := e^{-\alpha u}csch(u) = \frac{2e^{-\alpha u}}{e^u - e^{-u}}\,(u > 0),$$

where, $csch(u)$ is the hyperbolic cosecant function (cf. [25]), we obtain

$$e^{-\alpha xy}csch(xy) = \frac{2e^{-\alpha xy}}{e^{xy} - e^{-xy}},$$

$$e^{-\alpha x/y}csch(\frac{x}{y}) = \frac{2e^{-\alpha x/y}}{e^{x/y} - e^{-x/y}}$$

and for $\alpha > -1, \sigma > 1$,

$$\begin{aligned} K(\sigma,\alpha) \quad : &= \int_0^\infty e^{-\alpha u} csch(u) u^{\sigma-1} du \\ &= \int_0^\infty \frac{2u^{\sigma-1}e^{-\alpha u}}{e^u - e^{-u}} du = \int_0^\infty \frac{2u^{\sigma-1}e^{-(\alpha+1)u}}{1-e^{-2u}} du \\ &= 2\int_0^\infty u^{\sigma-1} \sum_{k=0}^\infty e^{-(2k+\alpha+1)u} du \\ &= 2\sum_{k=0}^\infty \int_0^\infty u^{\sigma-1} e^{-(2k+\alpha+1)u} du. \end{aligned}$$

Setting $v = (2k+\alpha+1)u$ in the above integral, we obtain

$$\begin{aligned} K(\sigma,\alpha) &= 2^{1-\sigma} \int_0^\infty v^{\sigma-1} e^{-v} dv \sum_{k=0}^\infty \frac{1}{(k+\frac{\alpha+1}{2})^\sigma} \\ &= 2^{1-\sigma}\Gamma(\sigma)\zeta(\sigma, \frac{\alpha+1}{2}) \in \mathbf{R}_+, \end{aligned} \tag{2.1}$$

where,

$$\Gamma(\sigma) := \int_0^\infty v^{\sigma-1} e^{-v} dv \, (\sigma > 0)$$

is the Gamma function, and

$$\zeta(\sigma, b) := \sum_{k=0}^\infty \frac{1}{(k+b)^\sigma} \, (Re\sigma > 1, b > 0)$$

is the extended Hurwitz-zeta function.
For $0 < b \le 1, \zeta(\sigma, b)$ is the Hurwitz-zeta function, and

$$\zeta(\sigma, 1) = \zeta(\sigma) = \sum_{k=1}^\infty \frac{1}{k^\sigma}$$

is the Riemann-zeta function (cf. [26], [24]).

In particular, for $\alpha = 0$, we find

$$\begin{aligned} K(\sigma, 0) &= 2^{1-\sigma}\Gamma(\sigma)\zeta(\sigma, \frac{1}{2}) = 2^{1-\sigma}\Gamma(\sigma)\sum_{k=0}^\infty \frac{1}{(k+\frac{1}{2})^\sigma} \\ &= 2\Gamma(\sigma)\sum_{k=0}^\infty \frac{1}{(2k+1)^\sigma} \\ &= 2\Gamma(\sigma)\left[\sum_{k=1}^\infty \frac{1}{k^\sigma} - \sum_{k=1}^\infty \frac{1}{(2k)^\sigma}\right] \\ &= 2\Gamma(\sigma)\left(1 - \frac{1}{2^\sigma}\right)\zeta(\sigma). \end{aligned}$$

Setting $\delta_0 := \frac{\sigma-1}{2} > 0$, we find $\sigma \pm \delta_0 \geq \sigma - \frac{\sigma-1}{2} = \frac{\sigma+1}{2} > 1$, and for $\alpha > -1$, we have

$$K(\sigma \pm \delta_0, \alpha) < \infty.$$

In the sequel we shall always assume that

$$0 < p < 1 \ (q < 0),\ \frac{1}{p} + \frac{1}{q} = 1,\ \alpha > -1,\ \sigma > 1,\ \delta_0 = \frac{\sigma - 1}{2} > 0 \ \textit{and}\ \sigma_1 \in \mathbf{R}.$$

For $n \in \mathbf{N} = \{1, 2, \cdots\}$, we define the following two expressions:

$$I_1 := \int_1^\infty \left(\int_0^1 e^{-\alpha xy} csch(xy) x^{\sigma + \frac{1}{pn} - 1} dx \right) y^{\sigma_1 - \frac{1}{qn} - 1} dy, \tag{2.2}$$

$$I_2 := \int_0^1 \left(\int_1^\infty e^{-\alpha xy} csch(xy) x^{\sigma - \frac{1}{pn} - 1} dx \right) y^{\sigma_1 + \frac{1}{qn} - 1} dy. \tag{2.3}$$

Setting $u = xy$ in (2.2) and (2.3), we have

$$\begin{aligned} I_1 &= \int_1^\infty \left[\int_0^y e^{-\alpha u} csch(u) \left(\frac{u}{y} \right)^{\sigma + \frac{1}{pn} - 1} \frac{1}{y} du \right] y^{\sigma_1 - \frac{1}{qn} - 1} dy \\ &= \int_1^\infty y^{(\sigma_1 - \sigma) - \frac{1}{n} - 1} \left(\int_0^y e^{-\alpha u} csch(u) u^{\sigma + \frac{1}{pn} - 1} du \right) dy, \end{aligned} \tag{2.4}$$

$$\begin{aligned} I_2 &= \int_0^1 \left[\int_y^\infty e^{-\alpha u} csch(u) \left(\frac{u}{y} \right)^{\sigma - \frac{1}{pn} - 1} \frac{1}{y} du \right] y^{\sigma_1 + \frac{1}{qn} - 1} dy \\ &= \int_0^1 y^{(\sigma_1 - \sigma) + \frac{1}{n} - 1} \left(\int_y^\infty e^{-\alpha u} csch(u) u^{\sigma - \frac{1}{pn} - 1} du \right) dy. \end{aligned} \tag{2.5}$$

Lemma 2.2. *If there exists a constant $M > 0$, such that for any non-negative measurable functions $f(x)$ and $g(y)$ in $(0, \infty)$, the following inequality*

$$\begin{aligned} I \ &:= \int_0^\infty \int_0^\infty e^{-\alpha xy} csch(xy) f(x) g(y) dx dy \\ &\geq M \left[\int_0^\infty x^{p(1-\sigma)-1} f^p(x) dx \right]^{\frac{1}{p}} \left[\int_0^\infty y^{q(1-\sigma_1)-1} g^q(y) dy \right]^{\frac{1}{q}} \end{aligned} \tag{2.6}$$

holds true, then we have $\sigma_1 = \sigma$.

Proof. If $\sigma_1 > \sigma$, then for $n > \frac{1}{\delta_0 p} (n \in \mathbf{N}, 0 < p < 1)$, we set the following two functions:

$$f_n(x) := \begin{cases} 0, 0 < x < 1 \\ x^{\sigma - \frac{1}{pn} - 1}, x \geq 1 \end{cases},$$

$$g_n(y) := \begin{cases} y^{\sigma_1 + \frac{1}{qn} - 1}, 0 < y \leq 1 \\ 0, y > 1 \end{cases}.$$

We find

$$\begin{aligned} J_2 \quad : \quad &= \left[\int_0^\infty x^{p(1-\sigma)-1} f_n^p(x)dx\right]^{\frac{1}{p}} \left[\int_0^\infty y^{q(1-\sigma_1)-1} g_n^q(y)dy\right]^{\frac{1}{q}} \\ &= \left(\int_1^\infty x^{-\frac{1}{n}-1}dx\right)^{\frac{1}{p}} \left(\int_0^1 y^{\frac{1}{n}-1}dy\right)^{\frac{1}{q}} = n. \end{aligned}$$

By (2.5), we have

$$\begin{aligned} I_2 &\leq \int_0^1 y^{(\sigma_1-\sigma)+\frac{1}{n}-1}dy \int_0^\infty e^{-\alpha u} csch(u) u^{\sigma-\frac{1}{pn}-1}du \\ &= \frac{1}{\sigma_1-\sigma+\frac{1}{n}} \left(\int_0^1 e^{-\alpha u} \operatorname{csc} h(u) u^{\sigma-\frac{1}{pn}-1}du \right. \\ &\quad \left. + \int_1^\infty e^{-\alpha u} \operatorname{csc} h(u) u^{\sigma-\frac{1}{pn}-1}du\right) \\ &\leq \frac{1}{\sigma_1-\sigma}\left(\int_0^1 e^{-\alpha u} csch(u) u^{\sigma-\delta_0-1}du + \int_1^\infty e^{-\alpha u} \operatorname{csc} h(u) u^{\sigma-1}du\right) \\ &\leq \frac{1}{\sigma_1-\sigma}\left(K(\sigma-\delta_0,\alpha)+K(\sigma,\alpha)\right), \end{aligned}$$

and then by (2.6), it follows that

$$\begin{aligned} &\frac{1}{\sigma_1-\sigma}\left(K(\sigma-\delta_0,\alpha)+K(\sigma,\alpha)\right) \\ \geq\ & I_2 = \int_0^\infty \int_0^\infty e^{-\alpha xy} \operatorname{csc} h(xy) f_n(x) g_n(y) dxdy \geq M J_2 = Mn. \quad (2.7) \end{aligned}$$

By (2.7), in view of $\sigma_1-\sigma>0$, $0 \leq K(\sigma-\delta_0,\alpha)+K(\sigma,\alpha)<\infty$, for $n\to\infty$, we deduce that

$$\infty > \frac{1}{\sigma_1-\sigma}\left(K(\sigma-\delta_0,\alpha)+K(\sigma,\alpha)\right) \geq \infty,$$

which is a contradiction.

If $\sigma_1<\sigma$, then for $n\in\mathbf{N}, n>\frac{1}{\delta_0 p}$, we set the following two functions:

$$\begin{aligned} \widetilde{f}_n(x) \quad : \quad &= \begin{cases} x^{\sigma+\frac{1}{pn}-1}, 0<x\leq 1 \\ 0, x>1 \end{cases}, \\ \widetilde{g}_n(y) \quad : \quad &= \begin{cases} 0, 0<y<1 \\ y^{\sigma_1-\frac{1}{qn}-1}, y\geq 1 \end{cases}. \end{aligned}$$

We obtain

$$\begin{aligned} \widetilde{J}_2 \quad : \quad &= \left[\int_0^\infty x^{p(1-\sigma)-1} \widetilde{f}_n^p(x)dx\right]^{\frac{1}{p}} \left[\int_0^\infty y^{q(1-\sigma_1)-1} \widetilde{g}_n^q(y)dy\right]^{\frac{1}{q}} \\ &= \left(\int_0^1 x^{\frac{1}{n}-1}dx\right)^{\frac{1}{p}} \left(\int_1^\infty y^{-\frac{1}{n}-1}dy\right)^{\frac{1}{q}} = n. \end{aligned}$$

By (2.4), we have

$$\begin{aligned} I_1 &\leq \int_1^{\infty} y^{(\sigma_1-\sigma)-\frac{1}{n}-1}dy \int_0^{\infty} e^{-\alpha u} csch(u) u^{\sigma+\frac{1}{pn}-1}du \\ &= \frac{1}{\sigma-\sigma_1+\frac{1}{n}}\left(\int_0^1 e^{-\alpha u} csch(u) u^{\sigma+\frac{1}{pn}-1}du \right. \\ &\quad \left. + \int_1^{\infty} e^{-\alpha u} csch(u) u^{\sigma+\frac{1}{pn}-1}du\right) \\ &\leq \frac{1}{\sigma-\sigma_1}\left(\int_0^1 e^{-\alpha u} csch(u) u^{\sigma-1}du + \int_1^{\infty} e^{-\alpha u} csch(u) u^{\sigma+\delta_0-1}du\right) \\ &\leq \frac{1}{\sigma-\sigma_1}\left(K(\sigma,\alpha)+K(\sigma+\delta_0,\alpha)\right), \end{aligned}$$

and then by (2.6), it follows that

$$\begin{aligned} &\frac{1}{\sigma-\sigma_1}\left(K(\sigma,\alpha)+K(\sigma+\delta_0,\alpha)\right) \\ \geq\ & I_1 = \int_0^{\infty}\int_0^{\infty} e^{-\alpha xy} csch(xy)\widetilde{f}_n(x)\widetilde{g}_n(y)dxdy \geq M\widetilde{J}_2 = Mn. \qquad (2.8) \end{aligned}$$

By (2.8), for $n\to\infty$, we get that

$$\infty > \frac{1}{\sigma-\sigma_1}\left(K(\sigma)+K(\sigma+\delta_0)\right) \geq \infty,$$

which is a contradiction.

Hence, we conclude that $\sigma_1=\sigma$.

The lemma is proved. □ □

For $\sigma_1=\sigma$, by Lemma 1, we still have

Lemma 2.3. *If there exists a constant $M>0$, such that for any non-negative measurable functions $f(x)$ and $g(y)$ in $(0,\infty)$, the following inequality*

$$\begin{aligned} I &:= \int_0^{\infty}\int_0^{\infty} e^{-\alpha xy} csch(xy) f(x)g(y)dxdy \\ &\geq M\left[\int_0^{\infty} x^{p(1-\sigma)-1}f^p(x)dx\right]^{\frac{1}{p}}\left[\int_0^{\infty} y^{q(1-\sigma)-1}g^q(y)dy\right]^{\frac{1}{q}} \qquad (2.9) \end{aligned}$$

holds true, then we have $M\leq K(\sigma,\alpha)$.

Proof. For $\sigma_1=\sigma$, in view of (2.6), we have

$$nM = MJ_2 \leq I_2\,.$$

Then we can apply (2.5) as follows:

$$\begin{aligned}
M &= \frac{1}{n}MJ_2 \leq \frac{1}{n}I_2 \\
&= \frac{1}{n}\int_0^1 y^{\frac{1}{n}-1}\left(\int_y^\infty e^{-\alpha u}csch(u)u^{\sigma-\frac{1}{pn}-1}du\right)dy \\
&= \frac{1}{n}\int_0^1 y^{\frac{1}{n}-1}\left(\int_y^1 e^{-\alpha u}\csc h(u)u^{\sigma-\frac{1}{pn}-1}du\right)dy \\
&\quad + \int_1^\infty e^{-\alpha u}\csc h(u)u^{\sigma-\frac{1}{pn}-1}du \\
&= \frac{1}{n}\int_0^1 \left(\int_0^u y^{\frac{1}{n}-1}dy\right)e^{-\alpha u}csch(u)u^{\sigma-\frac{1}{pn}-1}du \\
&\quad + \int_1^\infty e^{-\alpha u}\csc h(u)u^{\sigma-\frac{1}{pn}-1}du \\
&\leq \int_0^1 \csc h(u)u^{\sigma+\frac{1}{qn}-1}du + \int_1^\infty \csc h(u)u^{\sigma-1}du. \qquad (2.10)
\end{aligned}$$

For

$$n > \frac{1}{\delta_0|q|}(n \in \mathbf{N}),$$

we have

$$e^{-\alpha u}\csc h(u)u^{\sigma+\frac{1}{qn}-1} \leq e^{-\alpha u}csch(u)u^{\sigma-\delta_0-1}(0 < u \leq 1)$$

and

$$\int_0^1 e^{-\alpha u}csch(u)u^{\sigma-\delta_0-1}du \leq K(\sigma-\delta_0,\alpha) < \infty.$$

Therefore by (2.10) and Lebesgue's control convergence theorem (cf. [28]), we find

$$\begin{aligned}
M &\leq \lim_{n\to\infty}\left[\int_0^1 e^{-\alpha u}csch(u)u^{\sigma+\frac{1}{qn}-1}du + \int_1^\infty e^{-\alpha u}csch(u)u^{\sigma-1}du\right] \\
&= \int_0^1 \lim_{n\to\infty} e^{-\alpha u}csch(u)u^{\sigma+\frac{1}{qn}-1}du \\
&\quad + \int_1^\infty e^{-\alpha u}csch(u)u^{\sigma-1}du = K(\sigma).
\end{aligned}$$

The lemma is proved. □ □

3. Main results

Theorem 3.1. *Assuming that $M > 0$, the following conditions (i)-(iv) are equivalent:*

(i) For any $f(x) \geq 0$, satisfying

$$0 < \int_0^\infty x^{p(1-\sigma)-1}f^p(x)dx < \infty,$$

we have the following inequality:

$$J \;:\;= \left[\int_0^\infty y^{p\sigma_1-1}\left(\int_0^\infty e^{-\alpha xy} csch(xy)f(x)dx\right)^p dy\right]^{\frac{1}{p}}$$
$$> M\left[\int_0^\infty x^{p(1-\sigma)-1}f^p(x)dx\right]^{\frac{1}{p}}. \tag{3.1}$$

(ii) For any $g(y) \geq 0$, satisfying

$$0 < \int_0^\infty y^{q(1-\sigma_1)-1}g^q(y)dy < \infty,$$

we have the following inequality:

$$L \;:\;= \left[\int_0^\infty x^{q\sigma-1}\left(\int_0^\infty e^{-\alpha xy}\csc h(xy)g(y)dy\right)^q dx\right]^{\frac{1}{q}}$$
$$> M\left[\int_0^\infty y^{q(1-\sigma_1)-1}g^q(y)dy\right]^{\frac{1}{q}}. \tag{3.2}$$

(iii) For any $f(x) \geq 0$, satisfying

$$0 < \int_0^\infty x^{p(1-\sigma)-1}f^p(x)dx < \infty,$$

and $g(y) \geq 0$, satisfying

$$0 < \int_0^\infty y^{q(1-\sigma_1)-1}g^q(y)dy < \infty,$$

we have the following inequality:

$$I = \int_0^\infty\int_0^\infty e^{-\alpha xy} csch(xy)f(x)g(y)dxdy$$
$$> M\left[\int_0^\infty x^{p(1-\sigma)-1}f^p(x)dx\right]^{\frac{1}{p}}\left[\int_0^\infty y^{q(1-\sigma_1)-1}g^q(y)dy\right]^{\frac{1}{q}}. \tag{3.3}$$

(iv) $\sigma_1 = \sigma$, and $M \leq K(\sigma, \alpha)$.

Proof. $(i) \Rightarrow (iii)$. By the reverse Hölder inequality (cf. [27]), we have

$$I = \int_0^\infty\left(y^{\sigma_1-\frac{1}{p}}\int_0^\infty e^{-\alpha xy} csch(xy)f(x)dx\right)\left(y^{\frac{1}{p}-\sigma_1}g(y)\right)dy$$
$$\geq J\left[\int_0^\infty y^{q(1-\sigma_1)-1}g^q(y)dy\right]^{\frac{1}{q}}. \tag{3.4}$$

Then by (3.1), we have (3.3).

$(ii) \Rightarrow (iii)$. Again by the reverse Hölder inequality, we have

$$I = \int_0^\infty\left(y^{\frac{1}{q}-\sigma}f(x)\right)\left(x^{\sigma-\frac{1}{q}}\int_0^\infty e^{-\alpha xy} csch(xy)g(y)dy\right)dx$$
$$\geq \left[\int_0^\infty x^{p(1-\sigma)-1}f^p(x)dx\right]^{\frac{1}{p}}L. \tag{3.5}$$

Then by (3.2), we have (3.3).

$(iii) \Rightarrow (iv)$. By Lemma 1 and Lemma 2, we have $\sigma_1 = \sigma$, and $M \leq K(\sigma, \alpha)$.

$(iv) \Rightarrow (i)$. Setting $u = xy$, we obtain the following weight function: For $y > 0$,

$$\begin{aligned} \omega(\sigma, y) \quad : &= y^{\sigma} \int_0^{\infty} e^{-\alpha xy} csch(xy) x^{\sigma-1} dx \\ &= \int_0^{\infty} e^{-\alpha u} \operatorname{csc} h(u) u^{\sigma-1} du = K(\sigma, \alpha). \end{aligned} \tag{3.6}$$

By the reverse Hölder inequality with weight and (3.6), we have

$$\begin{aligned} & \left(\int_0^{\infty} e^{-\alpha xy} csch(xy) f(x) dx \right)^p \\ = & \left\{ \int_0^{\infty} e^{-\alpha xy} csch(xy) \left[\frac{y^{(\sigma-1)/p}}{x^{(\sigma-1)/q}} f(x) \right] \left[\frac{x^{(\sigma-1)/q}}{y^{(\sigma-1)/p}} \right] dx \right\}^p \\ \geq & \int_0^{\infty} e^{-\alpha xy} csch(xy) \frac{y^{\sigma-1}}{x^{(\sigma-1)p/q}} f^p(x) dx \\ & \times \left[\int_0^{\infty} e^{-\alpha xy} \operatorname{csc} h(xy) \frac{x^{\sigma-1}}{y^{(\sigma-1)q/p}} dx \right]^{p/q} \\ = & \left[\omega(\sigma, y) y^{q(1-\sigma)-1} \right]^{p-1} \int_0^{\infty} e^{-\alpha xy} csch(xy) \frac{y^{\sigma-1}}{x^{(\sigma-1)p/q}} f^p(x) dx \\ = & (K(\sigma, \alpha))^{p-1} y^{-p\sigma+1} \int_0^{\infty} e^{-\alpha xy} csch(xy) \frac{y^{\sigma-1}}{x^{(\sigma-1)p/q}} f^p(x) dx \end{aligned} \tag{3.7}$$

If (3.7) takes the form of equality for a $y \in (0, \infty)$, then there exist constants A and B, such that they are not all zero, and

$$A \frac{y^{\sigma-1}}{x^{(\sigma-1)p/q}} f^p(x) = B \frac{x^{\sigma-1}}{y^{(\sigma-1)q/p}} \; a.e. \; in \; \mathbf{R}_+$$

(cf. [27]). We suppose that $A \neq 0$ (otherwise $B = A = 0$). It follows that

$$x^{p(1-\sigma)-1} f^p(x) = y^{q(1-\sigma)} \frac{B}{Ax} \; a.e. \; in \; \mathbf{R}_+,$$

which contradicts the fact that

$$0 < \int_0^{\infty} x^{p(1-\sigma)-1} f^p(x) dx < \infty \, .$$

Hence, the middle of (3.7) takes the form of strict inequality.

For $\sigma_1 = \sigma$, by (3.7) with the above result and Fubini's theorem, we have

$$\begin{aligned} J &> (K(\sigma,\alpha))^{\frac{1}{q}} \left[\int_0^\infty \int_0^\infty e^{-\alpha xy} csch(xy) \frac{y^{\sigma-1}}{x^{(\sigma-1)p/q}} f^p(x) dx dy\right]^{\frac{1}{p}} \\ &= (K(\sigma,\alpha))^{\frac{1}{q}} \left\{\int_0^\infty \left[\int_0^\infty e^{-\alpha xy} \csc h(xy) \frac{y^{\sigma-1}}{x^{(\sigma-1)(p-1)}} dy\right] f^p(x) dx\right\}^{\frac{1}{p}} \\ &= (K(\sigma,\alpha))^{\frac{1}{q}} \left[\int_0^\infty \omega(\sigma,x) x^{p(1-\sigma)-1} f^p(x) dx\right]^{\frac{1}{p}} \\ &= K(\sigma,\alpha) \left[\int_0^\infty x^{p(1-\sigma)-1} f^p(x) dx\right]^{\frac{1}{p}}. \end{aligned} \tag{3.8}$$

Since

$$0 < M \leq K(\sigma,\alpha),$$

(3.1) follows.

$(iv) \Rightarrow (ii)$. Similarly to "$(iv) \Rightarrow (i)$", we obtain (3.2).

Therefore, the conditions (i), (ii), (iii) and (iv) are equivalent. □ □

For $\sigma_1 = \sigma$, we obtain the following theorem:

Theorem 3.2. *Assuming that $M > 0$, the following conditions (i)-(iv) are equivalent:*

(i) For any $f(x) \geq 0$, satisfying

$$0 < \int_0^\infty x^{p(1-\sigma)-1} f^p(x) dx < \infty,$$

we have the following inequality:

$$\begin{aligned} &\left[\int_0^\infty y^{p\sigma-1} \left(\int_0^\infty e^{-\alpha xy} csch(xy) f(x) dx\right)^p dy\right]^{\frac{1}{p}} \\ &> M \left[\int_0^\infty x^{p(1-\sigma)-1} f^p(x) dx\right]^{\frac{1}{p}}. \end{aligned} \tag{3.9}$$

(ii) For any $g(y) \geq 0$, satisfying

$$0 < \int_0^\infty y^{q(1-\sigma)-1} g^q(y) dy < \infty,$$

we have the following inequality:

$$\begin{aligned} &\left[\int_0^\infty x^{q\sigma-1} \left(\int_0^\infty e^{-\alpha xy} csch(xy) g(y) dy\right)^q dx\right]^{\frac{1}{q}} \\ &> M \left[\int_0^\infty y^{q(1-\sigma)-1} g^q(y) dy\right]^{\frac{1}{q}}. \end{aligned} \tag{3.10}$$

(iii) For any $f(x) \geq 0$, satisfying

$$0 < \int_0^\infty x^{p(1-\sigma)-1} f^p(x) dx < \infty,$$

and $g(y) \geq 0$, satisfying

$$0 < \int_0^\infty y^{q(1-\sigma)-1} g^q(y) dy < \infty,$$

we have the following inequality:

$$\int_0^\infty \int_0^\infty e^{-\alpha xy} csch(xy) f(x) g(y) dx dy$$
$$> M \left[\int_0^\infty x^{p(1-\sigma)-1} f^p(x) dx\right]^{\frac{1}{p}} \left[\int_0^\infty y^{q(1-\sigma)-1} g^q(y) dy\right]^{\frac{1}{q}}. \quad (3.11)$$

(iv) $M \leq K(\sigma, \alpha)$.

Moreover, if Condition (iv) follows, then the constant factor

$$M = K(\sigma, \alpha) = 2^{1-\sigma}\Gamma(\sigma)\zeta(\sigma, \frac{\alpha+1}{2})$$

in (3.9), (3.10) and (3.11) is the best possible.

Proof. For $\sigma_1 = \sigma$ in Theorem 1, we can prove that the conditions (i), (ii), (iii) and (iv) in Theorem 2 are equivalent. If there exists a constant $M \geq K(\sigma, \alpha)$, such that (3.11) is valid, then in view of $M \leq K(\sigma, \alpha)$, we can conclude that the constant factor $M = K(\sigma, \alpha)$ in (3.11) is the best possible.

The constant factor $K(\sigma, \alpha)$ in (3.9) ((3.10)) is still the best possible. Otherwise, by (3.4) ((3.5)) (for $\sigma_1 = \sigma$), we can conclude that the constant factor $M = K(\sigma, \alpha)$ in (3.11) is not the best possible.

The theorem is proved. □ □

4. Some particular cases

In particular, for $\sigma = \frac{1}{p}(> 1)$ in Theorem 2, we obtain the following corollary:

Corollary 4.1. *Assuming that $M > 0$, the following conditions (i)-(iv) are equivalent:*

(i) For any $f(x) \geq 0$, satisfying

$$0 < \int_0^\infty x^{p-2} f^p(x) dx < \infty,$$

we have the following inequality:

$$\left[\int_0^\infty \left(\int_0^\infty e^{-\alpha xy} csch(xy) f(x) dx\right)^p dy\right]^{\frac{1}{p}}$$
$$> M \left(\int_0^\infty x^{p-2} f^p(x) dx\right)^{\frac{1}{p}}. \quad (4.1)$$

(ii) For any $g(y) \geq 0$, satisfying

$$0 < \int_0^\infty g^q(y) dy < \infty,$$

we have the following inequality:

$$\left[\int_0^\infty x^{q-2}\left(\int_0^\infty e^{-\alpha xy} csch(xy) g(y)dy\right)^q dx\right]^{\frac{1}{q}}$$
$$> M\left(\int_0^\infty g^q(y)dy\right)^{\frac{1}{q}}. \tag{4.2}$$

(iii) For any $f(x) \geq 0$, satisfying

$$0 < \int_0^\infty x^{p-2} f^p(x)dx < \infty,$$

and $g(y) \geq 0$, satisfying

$$0 < \int_0^\infty g^q(y)dy < \infty,$$

we have the following inequality:

$$\int_0^\infty \int_0^\infty e^{-\alpha xy} csch(xy) f(x) g(y) dx dy$$
$$> M\left(\int_0^\infty x^{p-2} f^p(x)dx\right)^{\frac{1}{p}}\left(\int_0^\infty g^q(y)dy\right)^{\frac{1}{q}}; \tag{4.3}$$

(iv) The following inequality holds:

$$M \leq K(\frac{1}{p}, \alpha).$$

Moreover, if Condition (iv) follows, then the constant factor

$$M = K(\frac{1}{p}, \alpha) = 2^{\frac{1}{q}}\Gamma(\frac{1}{p})\zeta(\frac{1}{p}, \frac{\alpha+1}{2})$$

in (4.1), (4.2) and (4.3) is the best possible.

Setting

$$y = \frac{1}{Y}, \quad G(Y) = g(\frac{1}{Y})\frac{1}{Y^2}$$

in Theorem 1-2, then replacing Y $(G(Y))$ by y $(g(y))$, we have

Corollary 4.2. *Assuming that $M > 0$, the following Conditions (i)-(iv) are equivalent:*

(i) For any $f(x) \geq 0$, satisfying

$$0 < \int_0^\infty x^{p(1-\sigma)-1} f^p(x)dx < \infty,$$

we have the following inequality:

$$\left[\int_0^\infty y^{-p\sigma_1-1}\left(\int_0^\infty e^{\frac{\alpha x}{y}} csch(\frac{x}{y}) f(x)dx\right)^p dy\right]^{\frac{1}{p}}$$
$$> M\left[\int_0^\infty x^{p(1-\sigma)-1} f^p(x)dx\right]^{\frac{1}{p}}. \tag{4.4}$$

(ii) For any $g(y) \geq 0$, satisfying

$$0 < \int_0^\infty y^{q(1+\sigma_1)-1} g^q(y) dy < \infty,$$

we have the following inequality:

$$\left[\int_0^\infty x^{q\sigma-1} \left(\int_0^\infty e^{\frac{\alpha x}{y}} csch(\frac{x}{y}) g(y) dy\right)^q dx\right]^{\frac{1}{q}}$$
$$> M \left[\int_0^\infty y^{q(1+\sigma_1)-1} g^q(y) dy\right]^{\frac{1}{q}}. \quad (4.5)$$

(iii) For any $f(x) \geq 0$, satisfying

$$0 < \int_0^\infty x^{p(1-\sigma)-1} f^p(x) dx < \infty,$$

and $g(y) \geq 0$, satisfying

$$0 < \int_0^\infty y^{q(1+\sigma_1)-1} g^q(y) dy < \infty,$$

we have the following inequality:

$$\int_0^\infty \int_0^\infty e^{\frac{\alpha x}{y}} csch(\frac{x}{y}) f(x) g(y) dx dy$$
$$> M \left[\int_0^\infty x^{p(1-\sigma)-1} f^p(x) dx\right]^{\frac{1}{p}} \left[\int_0^\infty y^{q(1+\sigma_1)-1} g^q(y) dy\right]^{\frac{1}{q}}. \quad (4.6)$$

(iv) $\sigma_1 = \sigma$, and $M \leq K(\sigma, \alpha)$.

Moreover, if Condition (iv) follows, then the constant factor

$$M = K(\sigma, \alpha) = 2^{1-\sigma} \Gamma(\sigma) \zeta(\sigma, \frac{\alpha+1}{2})$$

in (4.4), (4.5) and (4.6) is the best possible.

In particular, for $\sigma = \frac{1}{p} (> 1)$ in Corollary 2, we obtain the following corollary:

Corollary 4.3. *Assuming that $M > 0$, the following conditions (i)-(iv) are equivalent:*

(i) For any $f(x) \geq 0$, satisfying

$$0 < \int_0^\infty x^{p-2} f^p(x) dx < \infty,$$

the following inequality holds true

$$\left[\int_0^\infty \frac{1}{y^2} \left(\int_0^\infty e^{\frac{\alpha x}{y}} csch(\frac{x}{y}) f(x) dx\right)^p dy\right]^{\frac{1}{p}}$$
$$> M \left(\int_0^\infty x^{p-2} f^p(x) dx\right)^{\frac{1}{p}}. \quad (4.7)$$

(ii) For any $g(y) \geq 0$, satisfying

$$0 < \int_0^\infty y^{2(q-1)} g^q(y) dy < \infty,$$

the following inequality holds true

$$\left[\int_0^\infty x^{q-2} \left(\int_0^\infty e^{\frac{\alpha x}{y}} csch(\frac{x}{y}) g(y) dy\right)^q dx\right]^{\frac{1}{q}} > M \left[\int_0^\infty y^{2(q-1)} g^q(y) dy\right]^{\frac{1}{q}}. \quad (4.8)$$

(iii) For any $f(x) \geq 0$, satisfying

$$0 < \int_0^\infty x^{p-2} f^p(x) dx < \infty,$$

and $g(y) \geq 0$, satisfying

$$0 < \int_0^\infty y^{2(q-1)} g^q(y) dy < \infty,$$

we have the the following inequality holds true

$$\int_0^\infty \int_0^\infty e^{\frac{\alpha x}{y}} csch(\frac{x}{y}) f(x) g(y) dx dy > M \left[\int_0^\infty x^{p-2} f^p(x) dx\right]^{\frac{1}{p}} \left[\int_0^\infty y^{2(q-1)} g^q(y) dy\right]^{\frac{1}{q}}. \quad (4.9)$$

(iv) The following inequality holds

$$M \leq K(\frac{1}{p}, \alpha).$$

Moreover, if Condition (iv) follows, then the constant factor

$$M = K(\frac{1}{p}, \alpha) = 2^{\frac{1}{q}} \Gamma(\frac{1}{p}) \zeta(\frac{1}{p}, \frac{\alpha+1}{2})$$

in (4.7), (4.8) and (4.9) is the best possible.

For $a = 0$ in Theorem 1, Theorem 2 and Corollary 2, we have the following two corollaries:

Corollary 4.4. *Assuming that $M > 0$, the following conditions (i)-(iv) are equivalent:*

(i) For any $f(x) \geq 0$, satisfying

$$0 < \int_0^\infty x^{p(1-\sigma)-1} f^p(x) dx < \infty,$$

the following inequality holds true

$$\left[\int_0^\infty y^{p\sigma_1-1}\left(\int_0^\infty csch(xy)f(x)dx\right)^p dy\right]^{\frac{1}{p}}$$
$$> M\left[\int_0^\infty x^{p(1-\sigma)-1}f^p(x)dx\right]^{\frac{1}{p}}. \quad (4.10)$$

(ii) For any $g(y) \geq 0$, satisfying

$$0 < \int_0^\infty y^{q(1-\sigma_1)-1}g^q(y)dy < \infty,$$

the following inequality holds true

$$\left[\int_0^\infty x^{q\sigma-1}\left(\int_0^\infty csch(xy)g(y)dy\right)^q dx\right]^{\frac{1}{q}}$$
$$> M\left[\int_0^\infty y^{q(1-\sigma_1)-1}g^q(y)dy\right]^{\frac{1}{q}}. \quad (4.11)$$

(iii) For any $f(x) \geq 0$, satisfying

$$0 < \int_0^\infty x^{p(1-\sigma)-1}f^p(x)dx < \infty,$$

and $g(y) \geq 0$, satisfying

$$0 < \int_0^\infty y^{q(1-\sigma_1)-1}g^q(y)dy < \infty,$$

the following inequality holds true

$$\int_0^\infty\int_0^\infty csch(xy)f(x)g(y)dxdy$$
$$> M\left[\int_0^\infty x^{p(1-\sigma)-1}f^p(x)dx\right]^{\frac{1}{p}}\left[\int_0^\infty y^{q(1-\sigma_1)-1}g^q(y)dy\right]^{\frac{1}{q}}. \quad (4.12)$$

(iv) The following holds true $\sigma_1 = \sigma$, and

$$M \leq K(\sigma, 0).$$

Moreover, if Condition (iv) is satisfied, then the constant factor

$$M = K(\sigma,0) = 2\Gamma(\sigma)\left(1 - \frac{1}{2^\sigma}\right)\zeta(\sigma)$$

in (4.10), (4.11) and (4.12) is the best possible.

Corollary 4.5. *Assuming that $M > 0$, the following Conditions (i)-(iv) are equivalent:*

(i) For any $f(x) \geq 0$, satisfying

$$0 < \int_0^\infty x^{p(1-\sigma)-1} f^p(x)dx < \infty,$$

the following inequality holds true

$$\left[\int_0^\infty y^{-p\sigma_1-1} \left(\int_0^\infty csch(\frac{x}{y})f(x)dx\right)^p dy\right]^{\frac{1}{p}} > M\left[\int_0^\infty x^{p(1-\sigma)-1} f^p(x)dx\right]^{\frac{1}{p}}. \tag{4.13}$$

(ii) For any $g(y) \geq 0$, satisfying

$$0 < \int_0^\infty y^{q(1+\sigma_1)-1} g^q(y)dy < \infty,$$

the following inequality holds true

$$\left[\int_0^\infty x^{q\sigma-1} \left(\int_0^\infty csch(\frac{x}{y})g(y)dy\right)^q dx\right]^{\frac{1}{q}} > M\left[\int_0^\infty y^{q(1+\sigma_1)-1} g^q(y)dy\right]^{\frac{1}{q}}. \tag{4.14}$$

(iii) For any $f(x) \geq 0$, satisfying

$$0 < \int_0^\infty x^{p(1-\sigma)-1} f^p(x)dx < \infty,$$

and $g(y) \geq 0$, satisfying

$$0 < \int_0^\infty y^{q(1+\sigma_1)-1} g^q(y)dy < \infty,$$

we have the following inequality holds true

$$\int_0^\infty \int_0^\infty \operatorname{csc} h(\frac{x}{y})f(x)g(y)dxdy > M\left[\int_0^\infty x^{p(1-\sigma)-1} f^p(x)dx\right]^{\frac{1}{p}} \left[\int_0^\infty y^{q(1+\sigma_1)-1} g^q(y)dy\right]^{\frac{1}{q}}. \tag{4.15}$$

(iv) The following holds true $\sigma_1 = \sigma$, and

$$M \leq K(\sigma, 0).$$

Moreover, if Condition (iv) is satisfied, then the constant factor

$$M = K(\sigma, 0) = 2\Gamma(\sigma)\left(1 - \frac{1}{2^\sigma}\right)\zeta(\sigma)$$

in (4.13), (4.14) and (4.15) is the best possible.

Acknowledgement.
B. C. Yang: This work is supported by the National Natural Science Foundation (Nos. 61370186, 61640222), and Appropriative Researching Fund for Professors

and Doctors, Guangdong University of Education (No. 2015ARF25). we are grateful for this help.

References

[1] V. Adiyasuren, T. Batbold, M. Krnić, On several new Hilbert-type inequalities involving means operators, Acta Mathematica Sinica (English Series), 29(8) (2013), 1493-1514.
[2] V. Adiyasuren, T. Batbold, M. Krnić, Half-discrete Hilbert-type inequalities with mean operators, the best constants and applications, Applied Mathematics and Computation 231 (2014), 148-159.
[3] G. H. Hardy, Note on a theorem of Hilbert concerning series of positive terms, Proceedings London Math. Soc., 1925, 23(2), Records of Proc. xlv-xlvi.
[4] I. Schur, Bernerkungen sur Theorie der beschrankten Billnearformen mit unendlich vielen Veranderlichen, J. Math., 1911,140,1-28.
[5] G. H. Hardy, J. E. Littlewood, G. Pólya, Inequalities, Cambridge University Press, Cambridge, USA, 1934.
[6] D. S. Mitrinović, J. E. Pečarić, A. M. Fink, Inequalities involving functions and their integrals and deivatives, Kluwer Academic, Boston, USA, 1991.
[7] B. C. Yang, On Hilbert's integral inequality, Journal of Mathematical Analysis and Applications, 1998, 220, 778-785.
[8] B. C. Yang, A note on Hilbert's integral inequality, Chinese Quarterly Journal of Mathematics, 1998, 13(4), 83-86.
[9] B. C. Yang, On an extension of Hilbert's integral inequality with some parameters, The Australian Journal of Mathematical Analysis and Applications, 2004, 1(1), Art.11,1-8.
[10] B. C. Yang, I. Brnetic, M. Krnić, J. E. Pecaric, Generalization of Hilbert and Hardy-Hilbert integral inequalities, Math. Ineq. and Appl., 2005, 8(2), 259-272.
[11] B. Yang, Q. Chen, A more accurate multidimensional Hardy-type inequality with a general homogeneous kernel, Journal of Mathematical Inequalities, 12(1) (2018), 113-128.
[12] M. Krnić, J. E. Pecaric, Hilbert's inequalities and their reverses, Publ. Math. Debrecen, 2005, 67(3-4), 315-331.
[13] Y. Hong. On Hardy-Hilbert integral inequalities with some parameters, J. Ineq. in Pure & Applied Math., 2005, 6(4), Art. 92: 1-10.
[14] B. Arpad, O. Choonghong, Best constant for certain multi linear integral operator, Journal of Inequalities and Applications, 2006, no. 28582.
[15] Y. J. Li, B. He, On inequalities of Hilbert's type, Bulletin of the Australian Mathematical Society, 2007, 76(1), 1-13.
[16] J. S. Xu, Hardy-Hilbert's Inequalities with two parameters, Advances in Mathematics, 2007, 36(2), 63-76.
[17] M. Krnić, M. Z. Gao, J. E. Pečarić, X. Gao, On the best constant in Hilbert's inequality. Math. Inequal. Appl, 2005, 8(2), 317-329.
[18] Y. Hong, On Hardy-type integral inequalities with some functions. Acta MathmaticaS inica, 2006, 49(1), 39-44.
[19] B. C. Yang, The norm of operator and Hilbert-type inequalities, Science Press, Beijing, China, 2009.
[20] B. C. Yang, On Hilbert-type integral inequalities and their operator expressions, Journal of Guangaong University of Education, 2013, 33(5), 1-17.
[21] B. C. Yang, Hilbert-type integral inequalities, Bentham Science Publishers Ltd., The United Emirates, 2009.
[22] Y. Hong, On the structure character of Hilbert's type integral inequality with homogeneous kernal and applications, Journal of Jilin University (Science Edition), 2017, 55(2), 189-194.
[23] M. Th. Rassias and B. C. Yang. Equivalent conditions of a Hardy-type integral inequality related to the extended Riemann zeta function. Adv. Oper. Theory, 2017, 2(3), 237–256.

[24] M. Th. Rassias. Problem-Solving and Selected Topics in Number Theory, Springer, 2011.
[25] Y. L. Pan, H. T. Wang and F. T. Wang. Complex functions. Science Press, Beijing, China, 2006.
[26] Z. Q. Wang and D. R. Guo. Introducution to special functions. Science Press, Beijing, China, 1979.
[27] J. C. Kuang, Applied inequalities. Shangdong Science and Technology Press, Jinan, China, 2004.
[28] J. C. Kuang, Real and functional analysis (Continuation)(second volume), Higher Education Press, Beijing, 2015.

[1]Institute of Mathematics, University of Zurich, CH-8057, Zurich, Switzerland & Moscow Institute of Physics and Technology, 141700 Dolgoprudny, Institutskiy per, d. 9, Russia & Institute for Advanced Study, Program in Interdisciplinary Studies, 1 Einstein Dr, Princeton, NJ 08540, USA.
E-mail address: michail.rassias@math.uzh.ch; rassias@ias.edu

[2]Department of Mathematics, Guangdong University of Education, Guangzhou, Guangdong 510303, P. R. China.
E-mail address: bcyang@gdei.edu.cn; bcyang818@163.com