ON AN EQUIVALENT PROPERTY OF A REVERSE HILBERT-TYPE INTEGRAL INEQUALITY RELATED TO THE EXTENDED HURWITZ-ZETA FUNCTION

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ABSTRACT. We study some equivalent conditions of a reverse Hilbert-type integral inequality with a particular non-homogeneous kernel and a best possible constant factor related to the extended Hurwitz-zeta function. Some equivalent conditions of a reverse Hilbert-type integral inequality with the particular homogeneous kernel are deduced. We also consider some particular cases.

1. Introduction

In 1925, Hardy [3] proved the following extension of Hilbert’s integral inequality (cf. [4]):
For $p > 1, \frac{1}{p} + \frac{1}{q} = 1, f(x), g(y) \geq 0,$

$$0 < \int_0^\infty f^p(x)dx < \infty \quad \text{and} \quad 0 < \int_0^\infty g^q(y)dy < \infty,$$

the following Hardy-Hilbert inequality holds true:

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y}dxdy < \frac{\pi}{\sin(\pi/p)} \left( \int_0^\infty f^p(x)dx \right)^{\frac{1}{p}} \left( \int_0^\infty g^q(y)dy \right)^{\frac{1}{q}},$$ (1.1)

with the best possible constant factor $\frac{\pi}{\sin(\pi/p)}$.
For $p = q = 2$, the inequality (1.1) reduces to Hilbert’s integral inequality, which is important in mathematical analysis and its applications (cf. [5], [6]).

In 1934, Hardy et al. extended the inequality (1.1) as follows:
If $p > 1, \frac{1}{p} + \frac{1}{q} = 1, k_1(x, y)$ is a nonnegative homogeneous function of degree $-1$, such that

$$k_p = \int_0^\infty k_1(u, 1)u^{\frac{-1}{p}} du \in \mathbb{R}_+ = (0, \infty),$$ (1.2)

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then we have the following Hardy-Hilbert-type integral inequality with the best possible constant $k_p$:

$$
\int_0^\infty \int_0^\infty k_1(x,y)f(x)g(y)\,dx\,dy
< k_p \left( \int_0^\infty f^p(x)\,dx \right)^{\frac{1}{p}} \left( \int_0^\infty g^q(y)\,dy \right)^{\frac{1}{q}}; \quad (1.3)
$$

for $0 < p < 1, \frac{1}{p} + \frac{1}{q} = 1$, the reverse of (1.3) follows (cf. [5], Theorem 319, Theorem 336). A Hilbert-type integral inequality with the non-homogeneous kernel was proved:

If $0 < p < 1, \frac{1}{p} + \frac{1}{q} = 1, h(u) > 0, \phi(\sigma) = \int_0^\infty h(u)u^{\sigma-1}du \in \mathbb{R}_+$, then

$$
\int_0^\infty \int_0^\infty h(xy)f(x)g(y)\,dx\,dy
< \phi\left( \frac{1}{p} \right)^{\frac{1}{p}} \left( \int_0^\infty x^{p-2}f^p(x)\,dx \right)^{\frac{1}{p}} \left( \int_0^\infty g^q(y)\,dy \right)^{\frac{1}{q}}, \quad (1.4)
$$

with the best possible constant factor $\phi\left( \frac{1}{p} \right)$ (cf. [5], Theorem 350).

In 1998, by introducing an independent parameter $\lambda > 0$, Yang proved an extension of Hilbert’s integral inequality with the kernel $\frac{1}{(x+y)^\lambda}$ (cf. [7], [8]). In 2004, by introducing another pair of conjugate exponents $(r, s)$, Yang [9] proved the following extension of inequality (1.1):

If $\lambda > 0, p, r > 1, \frac{1}{p} + \frac{1}{q} = \frac{1}{r} + \frac{1}{s} = 1, f(x), g(y) \geq 0$, satisfying

$$
0 < \int_0^\infty x^{p(1-\frac{1}{r})-1}f^p(x)\,dx < \infty
$$

and

$$
0 < \int_0^\infty y^{q(1-\frac{1}{s})-1}g^q(y)\,dy < \infty,
$$

then

$$
\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x^\lambda + y^\lambda} \,dx\,dy
< \frac{\pi}{\lambda \sin(\pi/r)} \left[ \int_0^\infty x^{p(1-\frac{1}{r})-1}f^p(x)\,dx \right]^\frac{1}{p} \left[ \int_0^\infty y^{q(1-\frac{1}{s})-1}g^q(y)\,dy \right]^\frac{1}{q}, \quad (1.5)
$$

with the best possible constant factor $\frac{\pi}{\lambda \sin(\pi/r)}$. In 2005, an extension of (1.1) with the kernel $\frac{1}{(x+y)^\lambda}$ and two pairs of conjugate exponents was proved in [10]. Krič et al. [12]-[18] provided some extensions and particular cases of (1.1), (1.3) and (1.5) with multi-parameters.

In 2009, Yang showed the following extension of (1.3) and (1.5) (cf. [19], [21]):

If $\lambda_1 + \lambda_2 = \lambda \in \mathbb{R} = (-\infty, \infty), k_\lambda(x, y)$ is a non-negative homogeneous function of degree $-\lambda$, satisfying

$$
k_\lambda(ux, uy) = u^{-\lambda}k_\lambda(x, y)(u, x, y > 0),$$
and

\[ k(\lambda_1) = \int_0^\infty k(\lambda)(x) dx \in \mathbb{R}_+ , \]

then for \( p > 1, \frac{1}{p} + \frac{1}{q} = 1 \), we have

\[
\int_0^\infty \int_0^\infty k(x,y) f(x)g(y) \, dx \, dy < k(\lambda_1) \left[ \int_0^\infty x^{p(1-\lambda_1)-1} f^p(x) \, dx \right]^{\frac{1}{p}} \left[ \int_0^\infty y^{q(1-\lambda_2)-1} g^q(y) \, dy \right]^{\frac{1}{q}} , \tag{1.6}
\]

with the best possible constant factor \( k(\lambda_1) \).

For \( 0 < p < 1, \frac{1}{p} + \frac{1}{q} = 1 \), the reverse of (1.6) follows. Additionally, the following extension of (1.4) was proved:

For \( p > 1, \frac{1}{p} + \frac{1}{q} = 1 \), we have

\[
\int_0^\infty \int_0^\infty h(x,y) f(x)g(y) \, dx \, dy < \phi(\sigma) \left[ \int_0^\infty x^{p(1-\sigma)-1} f^p(x) \, dx \right]^{\frac{1}{p}} \left[ \int_0^\infty y^{q(1-\sigma)-1} g^q(y) \, dy \right]^{\frac{1}{q}} , \tag{1.7}
\]

with the best possible constant factor \( \phi(\sigma) \).

For \( 0 < p < 1, \frac{1}{p} + \frac{1}{q} = 1 \), the reverse of (1.7) follows (cf. [20]).

Some equivalent inequalities of (1.6) are obtained in [21]. In 2013, Yang [20] studied as well the equivalency of (1.6) and (1.7). In 2017, in [22] and [23] some equivalent condition between a Hilbert-type integral inequality and the related parameters were investigated. For other closely related results the reader is also referred to [1], [2], [11], [23].

In this paper, by the use of techniques of real analysis and weight functions, we consider some equivalent conditions of a reverse of (1.7) in the particular kernel \( H(xy) = e^{-\alpha xy} \text{csch}(xy) \) where \( 0 < p < 1 \), with the best possible constant factor related to the extended Hurwitz-zeta function. Some equivalent conditions of the reverse of (1.6) for the particular kernel

\[ k_0(x,y) = e^{-\alpha x/y} \text{csc} h(\frac{x}{y}) \]

are deduced. We also consider some particular cases as corollaries.

2. An Example and Two Lemmas

**Example 2.1.** Setting

\[ H(u) := e^{-\alpha u} \text{csch}(u) = \frac{2e^{-\alpha u}}{e^u - e^{-u}} (u > 0) , \]

where, \( \text{csch}(u) \) is the hyperbolic cosecant function (cf. [25]), we obtain

\[ e^{-\alpha xy} \text{csch}(xy) = \frac{2e^{-\alpha xy}}{e^{xy} - e^{-xy}} , \]
\[
e^{-\alpha x/y} \text{csch} \left( \frac{x}{y} \right) = \frac{2e^{-\alpha x/y}}{e^{x/y} - e^{-x/y}}
\]
and for \( \alpha > -1, \sigma > 1 \),
\[
K(\sigma, \alpha) : = \int_0^\infty e^{-\alpha u} \text{csch}(u) u^{\sigma-1} du
\]
\[
= \int_0^\infty \frac{2u^{\sigma-1}e^{-\alpha u}}{e^{u} - e^{-u}} du = \int_0^\infty \frac{2u^{\sigma-1}e^{-(\alpha+1)u}}{1 - e^{-2u}} du
\]
\[
= 2 \int_0^\infty u^{\sigma-1} \sum_{k=0}^{\infty} e^{-(2k+\alpha+1)u} du
\]
\[
= 2 \sum_{k=0}^{\infty} \int_0^\infty u^{\sigma-1} e^{-(2k+\alpha+1)u} du.
\]
Setting \( v = (2k + \alpha + 1)u \) in the above integral, we obtain
\[
K(\sigma, \alpha) = 2^{1-\sigma} \int_0^\infty v^{\sigma-1} e^{-v} dv \sum_{k=0}^{\infty} \frac{1}{(k + \frac{\alpha+1}{2})^\sigma}
\]
\[
= 2^{1-\sigma} \Gamma(\sigma) \zeta(\sigma, \frac{\alpha+1}{2}) \in \mathbb{R}_+, \tag{2.1}
\]
where,
\[
\Gamma(\sigma) := \int_0^\infty v^{\sigma-1} e^{-v} dv \ (\sigma > 0)
\]
is the Gamma function, and
\[
\zeta(\sigma, b) := \sum_{k=0}^{\infty} \frac{1}{(k + b)^\sigma} \ (Re \sigma > 1, b > 0)
\]
is the extended Hurwitz-zeta function.
For \( 0 < b \leq 1 \), \( \zeta(\sigma, b) \) is the Hurwitz-zeta function, and
\[
\zeta(\sigma, 1) = \zeta(\sigma) = \sum_{k=1}^{\infty} \frac{1}{k^\sigma}
\]
is the Riemann-zeta function (cf. [26], [24]).
In particular, for \( \alpha = 0 \), we find
\[
K(\sigma, 0) = 2^{1-\sigma} \Gamma(\sigma) \zeta(\sigma, \frac{1}{2}) = 2^{1-\sigma} \Gamma(\sigma) \sum_{k=0}^{\infty} \frac{1}{(k + \frac{1}{2})^\sigma}
\]
\[
= 2 \Gamma(\sigma) \sum_{k=0}^{\infty} \frac{1}{(2k + 1)^\sigma}
\]
\[
= 2 \Gamma(\sigma) \left[ \sum_{k=1}^{\infty} \frac{1}{k^\sigma} - \sum_{k=1}^{\infty} \frac{1}{(2k)^\sigma} \right]
\]
\[
= 2 \Gamma(\sigma) \left( 1 - \frac{1}{2^\sigma} \right) \zeta(\sigma).
\]
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Setting \( \delta_0 := \frac{\sigma - 1}{2} > 0 \), we find \( \sigma \pm \delta_0 \geq \sigma - \frac{1}{2} + \frac{1}{2} = \sigma + 1 > 1 \), and for \( \alpha > -1 \), we have

\[ K(\sigma \pm \delta_0, \alpha) < \infty. \]

In the sequel we shall always assume that

\[ 0 < p < 1 \ (q < 0), \quad \frac{1}{p} + \frac{1}{q} = 1, \quad \alpha > -1, \quad \sigma > 1, \quad \delta_0 = \frac{\sigma - 1}{2} > 0 \quad \text{and} \quad \sigma_1 \in \mathbb{R}. \]

For \( n \in \mathbb{N} = \{1, 2, \cdots \} \), we define the following two expressions:

\[ I_1 := \int_1^\infty \left( \int_0^1 e^{-\alpha xy \text{csch}(xy)} x^{\sigma + \frac{1}{n} - 1} \, dx \right) y^{\sigma_1 - \frac{1}{n} - 1} \, dy, \quad (2.2) \]

\[ I_2 := \int_1^\infty \left( \int_0^1 e^{-\alpha xy \text{csch}(xy)} x^{\sigma - \frac{1}{n} - 1} \, dx \right) y^{\sigma_1 + \frac{1}{n} - 1} \, dy. \quad (2.3) \]

Setting \( u = xy \) in (2.2) and (2.3), we have

\[ I_1 = \int_1^\infty \left[ \int_0^1 e^{-\alpha u \text{csch}(u)} \left( \frac{u}{y} \right)^{\frac{\sigma - 1}{n} - 1} \frac{1}{y} \, du \right] y^{\sigma_1 - \frac{1}{n} - 1} \, dy, \]

\[ = \int_1^\infty y^{(\sigma_1 - \sigma) - \frac{1}{n} - 1} \left( \int_0^y e^{-\alpha u \text{csch}(u)} u^{\sigma + \frac{1}{n} - 1} \, du \right) \, dy, \quad (2.4) \]

\[ I_2 = \int_0^1 \left[ \int_y^\infty e^{-\alpha u \text{csch}(u)} \left( \frac{u}{y} \right)^{\frac{\sigma - 1}{n} - 1} \frac{1}{y} \, du \right] y^{\sigma_1 + \frac{1}{n} - 1} \, dy \]

\[ = \int_0^1 y^{(\sigma_1 - \sigma) + \frac{1}{n} - 1} \left( \int_y^\infty e^{-\alpha u \text{csch}(u)} u^{\sigma - \frac{1}{n} - 1} \, du \right) \, dy. \quad (2.5) \]

**Lemma 2.2.** If there exists a constant \( M > 0 \), such that for any non-negative measurable functions \( f(x) \) and \( g(y) \) in \((0, \infty)\), the following inequality

\[ I := \int_0^\infty \int_0^\infty e^{-\alpha xy \text{csch}(xy)} f(x)g(y) \, dx \, dy \]

\[ \geq M \left[ \int_0^\infty x^{p(1-\sigma)-1} f^p(x) \, dx \right]^{\frac{1}{p}} \left[ \int_0^\infty y^{q(1-\sigma_1)-1} g^q(y) \, dy \right]^{\frac{1}{q}} \quad (2.6) \]

holds true, then we have \( \sigma_1 = \sigma \).

**Proof.** If \( \sigma_1 > \sigma \), then for \( n > \frac{1}{\delta_0 p} \) \((n \in \mathbb{N}, 0 < p < 1)\), we set the following two functions:

\[ f_n(x) := \begin{cases} 0, & 0 < x < 1 \\ x^{\sigma + \frac{1}{n} - 1}, & x \geq 1 \end{cases}, \]

\[ g_n(y) := \begin{cases} y^{\sigma_1 + \frac{1}{n} - 1}, & 0 < y \leq 1 \\ 0, & y > 1 \end{cases}. \]
We find
\[
J_2 : = \left[ \int_0^\infty x^{p(1-\sigma)-1} f_n(x) dx \right]^{\frac{1}{p}} \left[ \int_0^\infty y^{q(1-\sigma_1)-1} g_n(y) dy \right]^{\frac{1}{q}}
\]
\[
= \left( \int_1^\infty x^{-\frac{1}{n}} dx \right)^{\frac{1}{p}} \left( \int_1^1 y^\frac{1}{n-1} dy \right)^{\frac{1}{q}} = n.
\]

By (2.5), we have
\[
I_2 \leq \int_0^1 y^{(\sigma_1-\sigma)} + \frac{1}{n} - 1 dy \int_0^\infty e^{-\alpha u} \text{csch}(u) u^{\sigma - \frac{1}{p}} du
\]
\[
= \frac{1}{\sigma_1 - \sigma + \frac{1}{n}} \left( \int_0^1 e^{-\alpha u} \csc h(u) u^{\sigma - \frac{1}{p}} du + \int_1^\infty e^{-\alpha u} \csc h(u) u^{\sigma - \frac{1}{p}} du \right)
\]
\[
\leq \frac{1}{\sigma_1 - \sigma} \left( \int_0^1 e^{-\alpha u} \text{csch}(u) u^{\sigma - \delta_0 - 1} du + \int_1^\infty e^{-\alpha u} \csc h(u) u^{\sigma - 1} du \right)
\]
\[
\leq \frac{1}{\sigma_1 - \sigma} (K(\sigma - \delta_0, \alpha) + K(\sigma, \alpha)),
\]
and then by (2.6), it follows that
\[
\frac{1}{\sigma_1 - \sigma} (K(\sigma - \delta_0, \alpha) + K(\sigma, \alpha))
\]
\[
\geq I_2 = \int_0^\infty \int_0^\infty e^{-\alpha xy} \csc h(xy) f_n(x) g_n(y) dxdy \geq MJ_2 = Mn. \tag{2.7}
\]

By (2.7), in view of \( \sigma_1 - \sigma > 0, 0 \leq K(\sigma - \delta_0, \alpha) + K(\sigma, \alpha) < \infty \), for \( n \to \infty \), we deduce that
\[
\infty > \frac{1}{\sigma_1 - \sigma} (K(\sigma - \delta_0, \alpha) + K(\sigma, \alpha)) \geq \infty,
\]
which is a contradiction.

If \( \sigma_1 < \sigma \), then for \( n \in \mathbb{N}, n > \frac{1}{\delta_0 p} \), we set the following two functions:
\[
\tilde{f}_n(x) : = \begin{cases} 
  x^{\sigma + \frac{1}{pn} - 1}, & 0 < x \leq 1 \\
  0, & x > 1
\end{cases},
\]
\[
\tilde{g}_n(y) : = \begin{cases} 
  0, & 0 < y < 1 \\
  y^{\sigma_1 - \frac{1}{qn} - 1}, & y \geq 1
\end{cases}.
\]

We obtain
\[
\tilde{J}_2 : = \left[ \int_0^\infty x^{p(1-\sigma)-1} \tilde{f}_n(x) dx \right]^{\frac{1}{p}} \left[ \int_0^\infty y^{q(1-\sigma_1)-1} \tilde{g}_n(y) dy \right]^{\frac{1}{q}}
\]
\[
= \left( \int_0^1 x^{-\frac{1}{n}} dx \right)^{\frac{1}{p}} \left( \int_0^\infty y^{-\frac{1}{n}} dy \right)^{\frac{1}{q}} = n.
\]
By (2.4), we have

\[ I_1 \leq \int_1^\infty y^{(\sigma_1 - \sigma) - \frac{1}{n} - 1} \, dy \int_0^\infty e^{-\alpha u \text{csch}(u)} u^{\sigma + \frac{1}{p} - 1} \, du \]

\[ = \frac{1}{\sigma - \sigma_1 + \frac{1}{n}} \left( \int_0^1 e^{-\alpha u \text{csch}(u)} u^{\sigma + \frac{1}{p} - 1} \, du \right. \]

\[ + \left. \int_1^\infty e^{-\alpha u \text{csch}(u)} u^{\sigma + \frac{1}{p} - 1} \, du \right) \]

\[ \leq \frac{1}{\sigma - \sigma_1} \left( \int_0^1 e^{-\alpha u \text{csch}(u)} u^{\sigma - 1} \, du + \int_1^\infty e^{-\alpha u \text{csch}(u)} u^{\sigma + \delta_0 - 1} \, du \right) \]

\[ \leq \frac{1}{\sigma - \sigma_1} (K(\sigma, \alpha) + K(\sigma + \delta_0, \alpha)), \]

and then by (2.6), it follows that

\[ \frac{1}{\sigma - \sigma_1} (K(\sigma, \alpha) + K(\sigma + \delta_0, \alpha)) \]

\[ \geq I_1 = \int_0^\infty \int_0^\infty e^{-\alpha xy \text{csch}(xy)} f_n(x) g_n(y) \, dx \, dy \geq M \tilde{J}_2 = Mn. \quad (2.8) \]

By (2.8), for \( n \to \infty \), we get that

\[ \infty > \frac{1}{\sigma - \sigma_1} (K(\sigma) + K(\sigma + \delta_0)) \geq \infty, \]

which is a contradiction.

Hence, we conclude that \( \sigma_1 = \sigma \).

The lemma is proved. \( \square \)

For \( \sigma_1 = \sigma \), by Lemma 1, we still have

**Lemma 2.3.** If there exists a constant \( M > 0 \), such that for any non-negative measurable functions \( f(x) \) and \( g(y) \) in \((0, \infty)\), the following inequality

\[ I = \int_0^\infty \int_0^\infty e^{-\alpha xy \text{csch}(xy)} f(x) g(y) \, dx \, dy \]

\[ \geq M \left[ \int_0^\infty x^{p(1-\sigma)-1} f^p(x) \, dx \right]^{\frac{1}{p}} \left[ \int_0^\infty y^{q(1-\sigma)-1} g^q(y) \, dy \right]^{\frac{1}{q}} \quad (2.9) \]

holds true, then we have \( M \leq K(\sigma, \alpha) \).

**Proof.** For \( \sigma_1 = \sigma \), in view of (2.6), we have

\[ nM = MJ_2 \leq I_2. \]
Then we can apply (2.5) as follows:

\[ M = \frac{1}{n} M J_2 \leq \frac{1}{n} I_2 \]

\[ = \frac{1}{n} \int_0^1 y^{\frac{1}{n}-1} \left( \int_y^\infty e^{-\alpha u} \csch(u) u^{\sigma - \frac{1}{p} - 1} du \right) dy \]

\[ = \frac{1}{n} \int_0^1 y^{\frac{1}{n}-1} \left( \int_y^1 e^{-\alpha u} \csc h(u) u^{\sigma - \frac{1}{p} - 1} du \right) dy \]

\[ + \int_1^\infty e^{-\alpha u} \csc h(u) u^{\sigma - \frac{1}{p} - 1} du \]

\[ = \frac{1}{n} \int_0^1 \left( \int_0^u y^{\frac{1}{n}-1} dy \right) e^{-\alpha u} \csch(u) u^{\sigma - \frac{1}{p} - 1} du \]

\[ + \int_1^\infty e^{-\alpha u} \csc h(u) u^{\sigma - \frac{1}{p} - 1} du \]

\[ \leq \int_0^1 \csc h(u) u^{\sigma + \frac{1}{q} - 1} du + \int_1^\infty \csc h(u) u^{\sigma - 1} du. \tag{2.10} \]

For

\[ n > \frac{1}{\delta_0|q|} (n \in \mathbb{N}), \]

we have

\[ e^{-\alpha u} \csc h(u) u^{\sigma + \frac{1}{q} - 1} \leq e^{-\alpha u} \csch(u) u^{\sigma - \delta_0 - 1} (0 < u \leq 1) \]

and

\[ \int_0^1 e^{-\alpha u} \csch(u) u^{\sigma - \delta_0 - 1} du \leq K(\sigma - \delta_0, \alpha) < \infty. \]

Therefore by (2.10) and Lebesgue’s control convergence theorem (cf. [28]), we find

\[ M \leq \lim_{n \to \infty} \left[ \int_0^1 e^{-\alpha u} \csch(u) u^{\sigma + \frac{1}{q} - 1} du + \int_1^\infty e^{-\alpha u} \csch(u) u^{\sigma - 1} du \right] \]

\[ = \int_0^1 \lim_{n \to \infty} e^{-\alpha u} \csch(u) u^{\sigma + \frac{1}{q} - 1} du \]

\[ + \int_1^\infty e^{-\alpha u} \csch(u) u^{\sigma - 1} du = K(\sigma). \]

The lemma is proved. \( \square \)

\[ \square \]

3. Main results

**Theorem 3.1.** Assuming that \( M > 0 \), the following conditions (i)-(iv) are equivalent:

(i) For any \( f(x) \geq 0 \), satisfying

\[ 0 < \int_0^\infty x^{p(1-\sigma) - 1} f^p(x) dx < \infty, \]
we have the following inequality:

\[
J := \left[ \int_{0}^{\infty} y^{\sigma_{1} - 1} \left( \int_{0}^{\infty} e^{-\alpha xy} \text{csch}(xy) f(x) dx \right)^{p} dy \right]^{\frac{1}{p}} > M \left[ \int_{0}^{\infty} x^{p(1-\sigma)} f^{p}(x) dx \right]^{\frac{1}{p}}.
\]

(3.1)

(iii) For any \( f(x) \geq 0 \), satisfying

\[
0 < \int_{0}^{\infty} x^{p(1-\sigma)} f^{p}(x) dx < \infty,
\]

and \( g(y) \geq 0 \), satisfying

\[
0 < \int_{0}^{\infty} y^{q(1-\sigma_{1}) - 1} g^{q}(y) dy < \infty,
\]

we have the following inequality:

\[
L := \left[ \int_{0}^{\infty} x^{q\sigma - 1} \left( \int_{0}^{\infty} e^{-\alpha xy} \csc h(xy) g(y) dy \right)^{q} dx \right]^{\frac{1}{q}} > M \left[ \int_{0}^{\infty} y^{q(1-\sigma_{1}) - 1} g^{q}(y) dy \right]^{\frac{1}{q}}.
\]

(3.2)

(iv) \( \sigma_{1} = \sigma \), and \( M \leq K(\sigma, \alpha) \).

**Proof.** (i) \( \Rightarrow \) (iii). By the reverse Hölder inequality (cf. [27]), we have

\[
I = \int_{0}^{\infty} \int_{0}^{\infty} e^{-\alpha xy} \text{csch}(xy) f(x) g(y) dx dy
\]

\[
> M \left[ \int_{0}^{\infty} x^{p(1-\sigma)} f^{p}(x) dx \right]^{\frac{1}{p}} \left[ \int_{0}^{\infty} y^{q(1-\sigma_{1}) - 1} g^{q}(y) dy \right]^{\frac{1}{q}}.
\]

(3.3)

Then by (3.1), we have (3.3).

(ii) \( \Rightarrow \) (iii). Again by the reverse Hölder inequality, we have

\[
I = \int_{0}^{\infty} \left( y^{\frac{1}{q} - \sigma} f(x) \right) \left( x^{\sigma - \frac{1}{q}} \int_{0}^{\infty} e^{-\alpha xy} \text{csch}(xy) g(y) dy \right) dx
\]

\[
> \left[ \int_{0}^{\infty} x^{p(1-\sigma)} f^{p}(x) dx \right]^{\frac{1}{p}} L.
\]

(3.5)
Then by (3.2), we have (3.3).

(iii) ⇒ (iv). By Lemma 1 and Lemma 2, we have $\sigma_1 = \sigma$, and $M \leq K(\sigma, \alpha)$.

(iv) ⇒ (i). Setting $u = xy$, we obtain the following weight function: For $y > 0$,

$$
\omega(\sigma, y) : = y^\sigma \int_0^\infty e^{-\alpha xy \operatorname{csch}(xy)} x^{\sigma-1} dx
= \int_0^\infty e^{-\alpha u \csc h(u)} u^{\sigma-1} du = K(\sigma, \alpha).
$$

By the reverse Hölder inequality with weight and (3.6), we have

$$
\left( \int_0^\infty e^{-\alpha xy \operatorname{csch}(xy)} f(x) dx \right)^p
= \left\{ \int_0^\infty e^{-\alpha xy \operatorname{csch}(xy)} \left[ \frac{y^{(\sigma-1)/p}}{x^{(\sigma-1)/q}} f(x) \right] \left[ \frac{x^{(\sigma-1)/q}}{y^{(\sigma-1)/p}} \right] dx \right\}^p
\geq \int_0^\infty e^{-\alpha xy \operatorname{csch}(xy)} \frac{y^{\sigma-1}}{x^{(\sigma-1)p/q}} f^p(x) dx
\times \left[ \int_0^\infty e^{-\alpha xy \csc h(y)} \frac{x^{\sigma-1}}{y^{(\sigma-1)q/p}} dx \right]^{p/q}
= \left[ \omega(\sigma, y)y^{q(1-\sigma)-1} \right]^{p-1} \int_0^\infty e^{-\alpha xy \operatorname{csch}(xy)} \frac{y^{\sigma-1}}{x^{(\sigma-1)p/q}} f^p(x) dx
= (K(\sigma, \alpha))^{p-1} y^{-p\sigma+1} \int_0^\infty e^{-\alpha xy \operatorname{csch}(xy)} \frac{y^{\sigma-1}}{x^{(\sigma-1)p/q}} f^p(x) dx \tag{3.7}
$$

If (3.7) takes the form of equality for a $y \in (0, \infty)$, then there exist constants $A$ and $B$, such that they are not all zero, and

$$
A \frac{y^{\sigma-1}}{x^{(\sigma-1)p/q}} f^p(x) = B \frac{x^{\sigma-1}}{y^{(\sigma-1)q/p}} a.e. \text{ in } \mathbb{R}_+
$$

(cf. [27]). We suppose that $A \neq 0$ (otherwise $B = A = 0$). It follows that

$$
x^{p(1-\sigma)-1} f^p(x) = y^{q(1-\sigma)} \frac{B}{Ax} a.e. \text{ in } \mathbb{R}_+,
$$

which contradicts the fact that

$$
0 < \int_0^\infty x^{p(1-\sigma)-1} f^p(x) dx < \infty.
$$

Hence, the middle of (3.7) takes the form of strict inequality.
For $\sigma_1 = \sigma$, by (3.7) with the above result and Fubini’s theorem, we have

\[
J > (K(\sigma, \alpha))^\frac{1}{q} \left[ \int_0^\infty \int_0^\infty e^{-\alpha xy} \text{csch}(xy) \frac{y^{\sigma-1}}{x^{(\sigma-1)p/q}} f^p(x) dx dy \right]^{\frac{1}{p}}
\]

\[
= (K(\sigma, \alpha))^\frac{1}{q} \left\{ \int_0^\infty \left[ \int_0^\infty e^{-\alpha xy} \text{csch}(xy) \frac{y^{\sigma-1}}{x^{(\sigma-1)p/(p-1)}} dx \right]^{\frac{1}{p}} \right\}^{\frac{1}{q}}
\]

\[
= (K(\sigma, \alpha))^\frac{1}{q} \left[ \int_0^\infty \omega(\sigma, x)^{p(1-\sigma)-1} f^p(x) dx \right]^{\frac{1}{p}}
\]

\[
= K(\sigma, \alpha)^{\frac{1}{q}} \left[ \int_0^\infty x^{p(1-\sigma)-1} f^p(x) dx \right]^{\frac{1}{p}}.
\]  

(3.8)

Since

\[
0 < M \leq K(\sigma, \alpha),
\]

(3.1) follows.

$(iv) \Rightarrow (ii)$. Similarly to “$(iv) \Rightarrow (i)$”, we obtain (3.2).

Therefore, the conditions (i), (ii), (iii) and (iv) are equivalent. $\square \quad \square$

For $\sigma_1 = \sigma$, we obtain the following theorem:

**Theorem 3.2.** Assuming that $M > 0$, the following conditions (i)-(iv) are equivalent:

(i) For any $f(x) \geq 0$, satisfying

\[
0 < \int_0^\infty x^{p(1-\sigma)-1} f^p(x) dx < \infty,
\]

we have the following inequality:

\[
\left[ \int_0^\infty y^\sigma (\int_0^\infty e^{-\alpha xy} \text{csch}(xy) f^p(x) dx)^{\frac{1}{p}} dy \right]^{\frac{1}{q}} > M \left[ \int_0^\infty x^{p(1-\sigma)-1} f^p(x) dx \right]^{\frac{1}{p}}.
\]  

(3.9)

(ii) For any $g(y) \geq 0$, satisfying

\[
0 < \int_0^\infty y^{q(1-\sigma)-1} g^q(y) dy < \infty,
\]

we have the following inequality:

\[
\left[ \int_0^\infty x^{q(1-\sigma)-1} (\int_0^\infty e^{-\alpha xy} \text{csch}(xy) g^q(y) dy)^{\frac{1}{q}} dx \right]^{\frac{1}{q}} > M \left[ \int_0^\infty y^{q(1-\sigma)-1} g^q(y) dy \right]^{\frac{1}{q}}.
\]  

(3.10)

(iii) For any $f(x) \geq 0$, satisfying

\[
0 < \int_0^\infty x^{p(1-\sigma)-1} f^p(x) dx < \infty,
\]
and \( g(y) \geq 0 \), satisfying
\[
0 < \int_{0}^{\infty} y^{q(1-\sigma)-1} g^q(y) dy < \infty,
\]
we have the following inequality:
\[
\int_{0}^{\infty} \int_{0}^{\infty} e^{-\alpha xy} \text{csch}(xy) f(x) g(y) dx dy > M \left[ \int_{0}^{\infty} x^{p(1-\sigma)-1} f^p(x) dx \right]^{\frac{1}{p}} \left[ \int_{0}^{\infty} y^{q(1-\sigma)-1} g^q(y) dy \right]^{\frac{1}{q}}.
\]
(3.11)

(iv) \( M \leq K(\sigma, \alpha) \).
Moreover, if Condition (iv) follows, then the constant factor
\[
M = K(\sigma, \alpha) = 2^{1-\sigma} \Gamma(\sigma) \zeta(\sigma, \frac{\alpha + 1}{2})
\]
in (3.9), (3.10) and (3.11) is the best possible.

**Proof.** For \( \sigma_1 = \sigma \) in Theorem 1, we can prove that the conditions (i), (ii), (iii) and (iv) in Theorem 2 are equivalent. If there exists a constant \( M \geq K(\sigma, \alpha) \), such that (3.11) is valid, then in view of \( M \leq K(\sigma, \alpha) \), we can conclude that the constant factor \( M = K(\sigma, \alpha) \) in (3.11) is the best possible.

The constant factor \( K(\sigma, \alpha) \) in (3.9) ((3.10)) is still the best possible. Otherwise, by (3.4) ((3.5)) (for \( \sigma_1 = \sigma \)), we can conclude that the constant factor \( M = K(\sigma, \alpha) \) in (3.11) is not the best possible.

The theorem is proved. \( \square \)

4. SOME PARTICULAR CASES

In particular, for \( \sigma = \frac{1}{p}(>1) \) in Theorem 2, we obtain the following corollary:

**Corollary 4.1.** Assuming that \( M > 0 \), the following conditions (i)-(iv) are equivalent:

(i) For any \( f(x) \geq 0 \), satisfying
\[
0 < \int_{0}^{\infty} x^{p-2} f^p(x) dx < \infty,
\]
we have the following inequality:
\[
\left[ \int_{0}^{\infty} \left( \int_{0}^{\infty} e^{-\alpha xy} \text{csch}(xy) f(x) dx \right)^p dy \right]^{\frac{1}{p}} > M \left( \int_{0}^{\infty} x^{p-2} f^p(x) dx \right)^{\frac{1}{p}}.
\]
(4.1)

(ii) For any \( g(y) \geq 0 \), satisfying
\[
0 < \int_{0}^{\infty} g^q(y) dy < \infty,
\]
we have the following inequality:

\[
\left[ \int_0^\infty x^{q-2} \left( \int_0^\infty e^{-\alpha xy} \text{csch}(xy)g(y)dy \right)^q dx \right]^{\frac{1}{q}} > M \left( \int_0^\infty g^q(y)dy \right)^{\frac{1}{q}}.
\]  

(4.2)

(iii) For any \( f(x) \geq 0 \), satisfying

\[
0 < \int_0^\infty x^{p-2} f^p(x)dx < \infty,
\]

and \( g(y) \geq 0 \), satisfying

\[
0 < \int_0^\infty g^q(y)dy < \infty,
\]

we have the following inequality:

\[
\int_0^\infty \int_0^\infty e^{-\alpha xy} \text{csch}(xy)f(x)g(y)dxdy > M \left( \int_0^\infty x^{p-2} f^p(x)dx \right)^{\frac{1}{p}} \left( \int_0^\infty g^q(y)dy \right)^{\frac{1}{q}};
\]

(4.3)

(iv) The following inequality holds:

\[
M \leq K\left( \frac{1}{p}, \alpha \right).
\]

Moreover, if Condition (iv) follows, then the constant factor

\[
M = K\left( \frac{1}{p}, \alpha \right) = 2^{\frac{1}{2}} \Gamma \left( \frac{1}{p} \right) \zeta \left( \frac{1}{p}, \frac{\alpha + 1}{2} \right)
\]

in (4.1), (4.2) and (4.3) is the best possible.

Setting

\[
y = \frac{1}{Y}, \quad G(Y) = g\left( \frac{1}{Y} \right) \frac{1}{Y^2}
\]

in Theorem 1-2, then replacing \( Y (G(Y)) \) by \( y (g(y)) \), we have

**Corollary 4.2.** Assuming that \( M > 0 \), the following Conditions (i)-(iv) are equivalent:

(i) For any \( f(x) \geq 0 \), satisfying

\[
0 < \int_0^\infty x^{p(1-\sigma)-1} f^p(x)dx < \infty,
\]

we have the following inequality:

\[
\left[ \int_0^\infty y^{-\sigma_1-1} \left( \int_0^\infty e^{\frac{\alpha x}{y}} \text{csch}\left( \frac{x}{y} \right)f(x)dx \right)^p dy \right]^{\frac{1}{p}} > M \left[ \int_0^\infty x^{p(1-\sigma)-1} f^p(x)dx \right]^{\frac{1}{p}}.
\]

(4.4)
(ii) For any \( g(y) \geq 0 \), satisfying
\[
0 < \int_0^\infty y^{q(1+\sigma_1)-1} g^q(y) dy < \infty,
\]
we have the following inequality:
\[
\left[ \int_0^\infty x^{q-1} \left( \int_0^\infty e^{\frac{\alpha}{y}} \text{csch} \left( \frac{x}{y} \right) g(y) dy \right)^q dx \right]^\frac{1}{q} > M \left[ \int_0^\infty y^{q(1+\sigma_1)-1} g^q(y) dy \right]^\frac{1}{q}.
\] (4.5)

(iii) For any \( f(x) \geq 0 \), satisfying
\[
0 < \int_0^\infty x^{p(1-\sigma)-1} f^p(x) dx < \infty,
\]
and \( g(y) \geq 0 \), satisfying
\[
0 < \int_0^\infty y^{q(1+\sigma_1)-1} g^q(y) dy < \infty,
\]
we have the following inequality:
\[
\int_0^\infty \int_0^\infty e^{\frac{\alpha x}{y}} \text{csch} \left( \frac{x}{y} \right) f(x) g(y) dx dy > M \left[ \int_0^\infty x^{p(1-\sigma)-1} f^p(x) dx \right]^\frac{1}{p} \left[ \int_0^\infty y^{q(1+\sigma_1)-1} g^q(y) dy \right]^\frac{1}{q}.
\] (4.6)

(iv) \( \sigma_1 = \sigma \), and \( M \leq K(\sigma, \alpha) \).
Moreover, if Condition (iv) follows, then the constant factor
\[
M = K(\sigma, \alpha) = 2^{1-\sigma} \Gamma(\sigma) \zeta(\sigma, \frac{\alpha + 1}{2})
\]
in (4.4), (4.5) and (4.6) is the best possible.

In particular, for \( \sigma = \frac{1}{p}(> 1) \) in Corollary 2, we obtain the following corollary:

**Corollary 4.3.** Assuming that \( M > 0 \), the following conditions (i)-(iv) are equivalent:

(i) For any \( f(x) \geq 0 \), satisfying
\[
0 < \int_0^\infty x^{p-2} f^p(x) dx < \infty,
\]
the following inequality holds true
\[
\left[ \int_0^\infty \frac{1}{y^2} \left( \int_0^\infty e^{\frac{\alpha x}{y}} \text{csch} \left( \frac{x}{y} \right) f(x) dx \right)^p dy \right]^\frac{1}{p} > M \left( \int_0^\infty x^{p-2} f^p(x) dx \right)^\frac{1}{p}.
\] (4.7)
(ii) For any \(g(y) \geq 0\), satisfying
\[
0 < \int_0^\infty y^{2(q-1)}g^q(y)dy < \infty,
\]
the following inequality holds true
\[
\left[\int_0^\infty x^{q-2} \left(\int_0^\infty e^{\frac{\alpha x}{y}}csch\left(\frac{x}{y}\right)g(y)dy\right)^q dx\right]^\frac{1}{q} > M \left[\int_0^\infty y^{2(q-1)}g^q(y)dy\right]^\frac{1}{q}.
\] (4.8)

(iii) For any \(f(x) \geq 0\), satisfying
\[
0 < \int_0^\infty x^{p-2}f^p(x)dx < \infty,
\]
and \(g(y) \geq 0\), satisfying
\[
0 < \int_0^\infty y^{2(q-1)}g^q(y)dy < \infty,
\]
we have the following inequality holds true
\[
\int_0^\infty \int_0^\infty e^{\frac{\alpha x}{y}}csch\left(\frac{x}{y}\right)f(x)g(y)dxdy > M \left[\int_0^\infty x^{p-2}f^p(x)dx\right]^\frac{1}{p} \left[\int_0^\infty y^{2(q-1)}g^q(y)dy\right]^\frac{1}{q}.
\] (4.9)

(iv) The following inequality holds
\[
M \leq K\left(\frac{1}{p}, \alpha\right).
\]
Moreover, if Condition (iv) follows, then the constant factor
\[
M = K\left(\frac{1}{p}, \alpha\right) = 2^\frac{3}{p} \Gamma\left(\frac{1}{p}\right)\zeta\left(\frac{1}{p}, \frac{\alpha + 1}{2}\right)
\]
in (4.7), (4.8) and (4.9) is the best possible.

For \(a = 0\) in Theorem 1, Theorem 2 and Corollary 2, we have the following two corollaries:

**Corollary 4.4.** Assuming that \(M > 0\), the following conditions (i)-(iv) are equivalent:

(i) For any \(f(x) \geq 0\), satisfying
\[
0 < \int_0^\infty x^{p(1-\sigma)-1}f^p(x)dx < \infty,
\]
the following inequality holds true
\[
\left[ \int_0^\infty y^{pq_1-1} \left( \int_0^\infty \text{csch}(xy)f(x)dx \right)^p \right]^{\frac{1}{p}} > M \left[ \int_0^\infty x^{p(1-\sigma)-1}f^p(x)dx \right]^{\frac{1}{p}}. \tag{4.10}
\]

(ii) For any \( g(y) \geq 0 \), satisfying
\[
0 < \int_0^\infty y^{q(1-\sigma_1)-1}g^q(y)dy < \infty,
\]
the following inequality holds true
\[
\left[ \int_0^\infty x^{q\sigma-1} \left( \int_0^\infty \text{csch}(xy)g(y)dy \right)^q \right]^{\frac{1}{q}} > M \left[ \int_0^\infty y^{q(1-\sigma_1)-1}g^q(y)dy \right]^{\frac{1}{q}}. \tag{4.11}
\]

(iii) For any \( f(x) \geq 0 \), satisfying
\[
0 < \int_0^\infty x^{p(1-\sigma)-1}f^p(x)dx < \infty,
\]
and \( g(y) \geq 0 \), satisfying
\[
0 < \int_0^\infty y^{q(1-\sigma_1)-1}g^q(y)dy < \infty,
\]
the following inequality holds true
\[
\int_0^\infty \int_0^\infty \text{csch}(xy)f(x)g(y)dxdy > M \left[ \int_0^\infty x^{p(1-\sigma)-1}f^p(x)dx \right]^{\frac{1}{p}} \left[ \int_0^\infty y^{q(1-\sigma_1)-1}g^q(y)dy \right]^{\frac{1}{q}}. \tag{4.12}
\]

(iv) The following holds true \( \sigma_1 = \sigma \), and
\[
M \leq K(\sigma, 0).
\]

Moreover, if Condition (iv) is satisfied, then the constant factor
\[
M = K(\sigma, 0) = 2\Gamma(\sigma) \left( 1 - \frac{1}{2\sigma} \right) \zeta(\sigma)
\]
in (4.10), (4.11) and (4.12) is the best possible.

**Corollary 4.5.** Assuming that \( M > 0 \), the following Conditions (i)-(iv) are equivalent:
(i) For any \( f(x) \geq 0 \), satisfying

\[
0 < \int_{0}^{\infty} x^{p(1-\sigma)-1} f^p(x) dx < \infty,
\]

the following inequality holds true

\[
\left[ \int_{0}^{\infty} y^{-p\sigma_1-1} \left( \int_{0}^{\infty} \text{csch} \left( \frac{x}{y} \right) f(x) dx \right)^p dy \right]^{\frac{1}{p}} > M \left[ \int_{0}^{\infty} x^{p(1-\sigma)-1} f^p(x) dx \right]^{\frac{1}{p}}. 
\]

(4.13)

(ii) For any \( g(y) \geq 0 \), satisfying

\[
0 < \int_{0}^{\infty} y^{q(1+\sigma_1)-1} g^q(y) dy < \infty,
\]

the following inequality holds true

\[
\left[ \int_{0}^{\infty} x^{q\sigma-1} \left( \int_{0}^{\infty} \text{csch} \left( \frac{x}{y} \right) g(y) dy \right)^q dx \right]^{\frac{1}{q}} > M \left[ \int_{0}^{\infty} y^{q(1+\sigma_1)-1} g^q(y) dy \right]^{\frac{1}{q}}. 
\]

(4.14)

(iii) For any \( f(x) \geq 0 \), satisfying

\[
0 < \int_{0}^{\infty} x^{p(1-\sigma)-1} f^p(x) dx < \infty,
\]

and \( g(y) \geq 0 \), satisfying

\[
0 < \int_{0}^{\infty} y^{q(1+\sigma_1)-1} g^q(y) dy < \infty,
\]

we have the following inequality holds true

\[
\int_{0}^{\infty} \int_{0}^{\infty} \text{csch} \left( \frac{x}{y} \right) f(x) g(y) dx dy > M \left[ \int_{0}^{\infty} x^{p(1-\sigma)-1} f^p(x) dx \right]^{\frac{1}{2}} \left[ \int_{0}^{\infty} y^{q(1+\sigma_1)-1} g^q(y) dy \right]^{\frac{1}{2}}. 
\]

(4.15)

(iv) The following holds true \( \sigma_1 = \sigma \), and

\[
M \leq K(\sigma, 0).
\]

Moreover, if Condition (iv) is satisfied, then the constant factor

\[
M = K(\sigma, 0) = 2\Gamma(\sigma) \left( 1 - \frac{1}{2\pi} \right) \zeta(\sigma)
\]

in (4.13), (4.14) and (4.15) is the best possible.

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