ASYMPTOTICS FOR MOMENTS OF CERTAIN COTANGENT SUMS FOR ARBITRARY EXPONENTS

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Abstract. In this paper we extend a result on the asymptotics of moments of certain cotangent sums associated to the Estermann and Riemann zeta functions for arbitrary positive real exponents.
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1. Introduction

The authors in joint work [8] and the second author in his thesis [12], investigated the distribution of cotangent sums

$$c_0\left(\frac{r}{b}\right) = -\sum_{m=1}^{b-1} \frac{m}{b} \cot\left(\frac{\pi m r}{b}\right)$$

as r ranges over the set

$$\{r \ : \ (r,b)=1,\ A_0 b \leq r \leq A_1 b\}\ ,$$

where A_0, A_1 are fixed with $1/2 < A_0 < A_1 < 1$ and b tends to infinity.
The reader is referred to [8], [12] for a demonstration of the significance of these sums by an exhibition of their relation to other important functions in Mathematics, such as the Estermann and the Riemann zeta functions, as well as their connections to major open problems, such as the Riemann Hypothesis.
In subsequent work [9] the authors considered the moments defined by

$$H_k = \lim_{b\to+\infty} \phi(b)^{-1} b^{-2k} (A_1 - A_0)^{-1} \sum_{\substack{A_0 b \leq r \leq A_1 b \\ (r,b)=1}} c_0\left(\frac{r}{b}\right)^{2k},\ k \in \mathbb{N},$$

where $\phi(\cdot)$ denoted the Euler phi-function. They could show that

$$H_k = \int_0^1 \left(\frac{g(x)}{\pi}\right)^{2k} dx\ ,$$

where

$$g(x) = \sum_{l\geq 1} \frac{1-2\{lx\}}{l}\ ,$$

a function that has been investigated by de la Bretèche and Tenenbaum [5, 6], as well as Balazard and Martin [2, 3]. Bettin [4] could replace the interval $(1/2, 1)$ for A_0, A_1 by the interval $(0, 1)$.

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Improving on a result on the order of magnitude of $\int_0^1 g(x)^{2k}dx$ obtained in the paper [9], the authors could obtain the following asymptotics (Theorem 1.1 of [10]):

Let $K \in \mathbb{N}$. There is an absolute constant $C > 0$, such that

$$\int_0^1 |g(x)|^K dx = \frac{e^\gamma}{\pi}\,\Gamma(K+1)(1+O(\exp(-CK))), \tag{1.1}$$

for $K \to \infty$, where γ is the Euler-Mascheroni constant.

The authors are thankful to Goubi Mouloud for the information on the value e^γ/π of the constant.

In this paper we extend the result (1.1) to all positive real values for the exponent K.

Theorem 1.1. *Let $K \in \mathbb{R}$, $K > 0$. There is an absolute constant $C > 0$, such that*

$$\int_0^1 |g(x)|^K dx = \frac{e^\gamma}{\pi}\,\Gamma(K+1)(1+O(\exp(-CK))),$$

for $K \to \infty$, where γ is the Euler-Mascheroni constant.

2. Overview and preliminary results

Like in previous papers, a crucial role is played by the relation of $g(x)$ to Wilton's function, established by Balazard and Martin [3] and results about operators related to continued fraction expansions due to Marmi, Moussa and Yoccoz [11].

We recall some fundamental definitions and results from [11]. For the proofs of Lemmas 2.2, 2.4, 2.6 of the present paper, see [9].

Definition 2.1. *Let $X = (0,1)\setminus\mathbb{Q}$. Let $\alpha(x) = \{1/x\}$ for $x \in X$. The iterates α_k of α are defined by $\alpha_0(x) = x$ and*

$$\alpha_k(x) = \alpha(\alpha_{k-1}(x)),\ \text{for } k > 1.$$

Lemma 2.2. *Let $x \in X$ and let*

$$x = [a_0(x); a_1(x), \ldots, a_k(x), \ldots]$$

be the continued fraction expansion of x. We define the partial quotient of $p_k(x)$, $q_k(x)$:

$$\frac{p_k(x)}{q_k(x)} := [a_0(x); a_1(x), \ldots, a_k(x)],\ \text{where, } (p_k(x), q_k(x)) = 1\,.$$

Then we have

$$a_k(x) = \left\lfloor \frac{1}{\alpha_{k-1}(x)} \right\rfloor,$$

$$p_{k+1} = a_{k+1}p_k + p_{k-1}$$

and

$$q_{k+1} = a_{k+1}q_k + q_{k-1}\,.$$

Definition 2.3. *Let $x \in X$. Let also*

$$\beta_k(x) := \alpha_0(x)\alpha_1(x)\cdots\alpha_k(x),\ \beta_{-1}(x) = 1$$

$$\gamma_k(x) := \beta_{k-1}(x)\log\frac{1}{\alpha_k(x)},\ \text{where } k \geq 0,$$

so that $\gamma_0(x) := \log(1/x)$.
The number x *is called a* ***Wilton number*** *if the series*

$$\sum_{k\geq 0}(-1)^k\gamma_k(x)$$

converges.
Wilton's function $\mathcal{W}(x)$ *is defined by*

$$\mathcal{W}(x) = \sum_{k\geq 0}(-1)^k\gamma_k(x)$$

for each Wilton number $x \in (0,1)$.

Lemma 2.4. *A number* $x \in X$ *is a Wilton number if and only if* $\alpha(x)$ *is a Wilton number. In this case we have:*

$$\mathcal{W}(x) = \log\frac{1}{x} - x\mathcal{W}(\alpha(x)).$$

Definition 2.5. *Let* $p > 1$ *and* $T : L^p \to L^p$ *be defined by*

$$Tf(x) := xf(\alpha(x)).$$

The measure m *is defined by*

$$m(\mathcal{E}) := \frac{1}{\log 2}\int_{\mathcal{E}}\frac{dx}{1+x},$$

where $\mathcal{E}$ *is any measurable subset of* $(0,1)$.

Lemma 2.6. *Let* $p > 1$, $n \in \mathbb{N}$.
(i) The measure m *is invariant with respect to the map* α, *i.e.*

$$m(\alpha(\mathcal{E})) = m(\mathcal{E})\ ,$$

for all measurable subsets of $\mathcal{E} \subset (0,1)$.
(ii) For $f \in L^p$ *we have*

$$\int_0^1 |T^n f(x)|^p dm(x) \leq g^{(n-1)p}\int_0^1 |f(x)|^p dm(x),$$

where

$$g := \frac{\sqrt{5}-1}{2} < 1.$$

Definition 2.7. *For* $n \in \mathbb{N}$, $x \in X$, *we define*

$$\mathcal{L}(x,n) := \sum_{v=0}^{n}(-1)^v(T^v l)(x),$$

where $l(x) := \log\left(\frac{1}{x}\right)$,

$$D(x,n) := \mathcal{L}(x,n) - l(x).$$

We recall the following definitions from [10].

Definition 2.8. *For* $\lambda \geq 0$*, we set*

$$A(\lambda) := \int_0^\infty \{t\}\{\lambda t\} \frac{dt}{t^2}\,,$$
$$F(x) := \frac{x+1}{2} A(1) - A(x) - \frac{x}{2}\log x\,,$$
$$H(x) := 2\sum_{j\geq 0}(-1)^j \beta_{j-1}(x) F(\alpha_j(x))\,,$$
$$B_1(t) := t - \lfloor t \rfloor - 1/2,\ \textit{the first Bernoulli function}\,,$$
$$B_2(t) := \{t\}^2 - \{t\} + 1/6,\ (t \in \mathbb{R})\ \textit{the second Bernoulli function}\,.$$

For $\lambda \in \mathbb{R}$*, let*

$$\Phi_2(\lambda) := \sum_{n\geq 1} \frac{B_2(n\lambda)}{n^2}\,.$$

Lemma 2.9. *It holds*

$$A(\lambda) = \frac{\lambda}{2}\log\frac{1}{\lambda} + \frac{1+A(1)}{2}\,\lambda + O(\lambda^2), \quad \textit{as } \lambda \to 0\,.$$

Proof. By [3], Proposition 31, formula (74), we have:

$$A(\lambda) = \frac{\lambda}{2}\log\frac{1}{\lambda} + \frac{1+A(1)}{2}\,\lambda + \frac{\lambda^2}{2}\Phi_2\left(\frac{1}{\lambda}\right) - \int_{1/\lambda}^\infty \Phi_2(t)\frac{dt}{t^3}\,.$$

From Definition 2.8, it follows that $\Phi_2(t)$ is bounded. Therefore

$$\frac{\lambda^2}{2}\Phi_2\left(\frac{1}{\lambda}\right) = O(\lambda^2)$$

and

$$\int_{1/\lambda}^\infty \Phi_2(t)\frac{dt}{t^3} = O(\lambda^2).$$

□

Lemma 2.10. *We have*

$$g(x) = l(x) + D(x,n) + H(x) + (-1)^{n+1}T^{n+1}\mathcal{W}(x).$$

Proof. From formula (3) of [9] we have:

$$\mathcal{W}(x) = \mathcal{L}(x,n) + (-1)^{n+1}T^{n+1}\mathcal{W}(x). \tag{2.1}$$

In [3] the function Φ_1 is defined by

$$\Phi_1(t) := \sum_{n\geq 1}\frac{B_1(nt)}{n} = \sum_{n\geq 1}\frac{\{nt\}-1/2}{n}\,. \tag{2.1}$$

Thus we have

$$g(x) = -2\Phi_1(x)\,. \tag{2.2}$$

By Proposition (2) of [3] we obtain

$$\Phi_1(x) = -\frac{1}{2}\mathcal{W}(x) - \frac{1}{2}H(x) \tag{2.3}$$

almost everywhere.

The proof of Lemma 2.10 follows now from (2.1), (2.2), (2.3) and Definition 2.7 □

3. Proof of Theorem 1.1

Definition 3.1. *Let $d, h \in \mathbb{N}_0$, $h \geq 1$, $u, v \in (0, \infty)$. Then we define*

$$\mathcal{J}(d,h,u,v) := \{x \in X \,:\, T^d l(x) \geq u \text{ and } T^{d+h} l(x) \geq v\}\,.$$

Lemma 3.2. *We have*

$$m(\mathcal{J}(d,h,u,v)) \leq 2\exp\left(-2^{\frac{h-2}{2}} v \exp\left(2^{\frac{d-2}{2}} u\right)\right)$$

Proof. This is Lemma 2.13 of [9]. □

We recall the following definition from [9].

Definition 3.3. *(Definition 2.14 of [9])*
Let $L \in \mathbb{N}$. We set $j_0 := L - \left\lfloor \frac{L}{100} \right\rfloor$, $C_1 := 1/400$. For $j \in \mathbb{Z}$, $j \leq j_0$, we define the intervals:

$$I(L,j) := \left(x^{(j-1)}, x^{(j)}\right), \text{ where } x^{(j)} := \exp(-L+j).$$

For $v \in \mathbb{N}_0$, we set

$$a(L,v) := \exp(-C_1 L + v)$$

$$\mathcal{T}(L,j,0) := \{x \in I(L,j) \cap X \,:\, |D(x,n)| \leq \exp(-C_1 L)\}\,,$$

and for $v \in \mathbb{N}$, we set

$$\mathcal{T}(L,j,v) := \{x \in I(L,j) \cap X \,:\, a(L,v-1) \leq |D(x,n)| \leq a(L,v)\}\,.$$

For $v, h \in \mathbb{Z}$, $h \geq 0$, we set

$$U(L,j,v,h) := \{x \in \mathcal{T}(L,j,v) \,:\, T^h l(x) \geq 2^{-h} a(L,v-1)\}\,.$$

Lemma 3.4. *There are constants $C_2, C_3 > 0$, such that for $v \geq 1$, we have*

$$m(\mathcal{T}(L,j,v)) \leq C_2 \exp\left(-C_3 \exp\left(-C_1 L + v - 1 + \frac{1}{2}(L-j)\right)\right)\,.$$

Proof. This is lemma 2.15 of [9]. □

Definition 3.5. *Let $L_0 := \lfloor K \rfloor + 1$. For $j \in \mathbb{Z}$, $j \leq j_0$ we define:*

$$\mathcal{E}_1(K,j,n) := \left\{x \in I(L_0,j) \,:\, |D(x,n)| \geq \exp\left(-\frac{C_1}{2}K\right)\right\}$$

$$\mathcal{E}_2(K,j,n) := \left\{x \in I(L_0,j) \,:\, |T^{n+1}\mathcal{W}(x)| \geq \exp\left(-\frac{C_1}{2}K\right)\right\}.$$

Lemma 3.6. *For sufficiently large K we have:*

$$m(\mathcal{E}_1(K,j,n)) \leq |I(L_0,j)| \exp(-K)\,.$$

Proof. We have

$$|D(x,n)| \geq \exp\left(-\frac{C_1}{2}K\right)$$

Therefore

$$x \in \bigcup_{v \geq \frac{C_1}{3}K} \mathcal{T}(L_0,j,v)$$

and thus by Lemma 3.4 we have

$$m(\mathcal{E}_1(K,j,n)) \leq C_2 \sum_{v \geq \frac{C_1}{3}K} \exp\left(-C_3 \exp\left(-C_1 L_0 + v - 1 + \frac{1}{2}(L-j)\right)\right) \leq |I_0(L_0,j)| \exp(-K).$$

□

Definition 3.7. *For $w \in \mathbb{N}_0$ we set*

$$\mathcal{V}(K,j,w,n) := \{x \in I(L_0,j) : l(x)\exp\left(-\frac{C_1}{2}K + w\right) \le |T^{(n+1)}\mathcal{W}(x)| \le l(x)\exp\left(-\frac{C_1}{2}K + w + 1\right)$$

$$\mathcal{Z}(K,j,v,w,n) := \mathcal{T}(L_0,j,v) \cap \mathcal{V}(K,j,w,n)\,.$$

Lemma 3.8. *Here and in the sequel we assume that $n \ge n_0(K)$, where $n_0(K)$ is chosen sufficiently large. It holds*

$$m(\mathcal{V}(K,j,w,n)) \le \exp(-2w)(L_0 - j - H + w)^{-1}\exp(-\exp(4K))\,,$$

where

$$H := \sup_{x\in(0,1)} |H(x)|.$$

Proof. By Lemma 2.6 we have

$$\begin{aligned} m(\mathcal{V}(K,j,w,n))(L_0 - j - H + w)^2\exp(-C_1K + 2w) &\le \int_{\mathcal{V}(K,j,w,n)} |T^{n+1}\mathcal{W}(x)|^2 dm(x) \\ &\le g^{2(n-1)}\int_0^1 |\mathcal{W}(x)|^2 dm(x). \end{aligned}$$

Thus

$$m(\mathcal{V}(K,j,w,n)) \le g^{2(n-1)}\left(\int_0^1 \mathcal{W}(x)^2 dm(x)\right)\exp\left(\frac{C_1}{2}K - 2w\right)(L_0 - j - H + w)^{-2}.$$

The result of Lemma 3.8 follows by choosing n sufficiently large. $\square$

Lemma 3.9. *We have for $n \ge n_0(K)$:*

$$\int_{\mathcal{E}_1(K,j,n)} |g(x)|^K dx \le |I(K,j)|\exp(-K).$$

Proof. We have

$$\mathcal{E}_1(K,j,n) \subseteq \bigcup_{v\ge 1} \mathcal{T}(K,j,v)$$

and therefore by Definition 3.3:

$$\begin{aligned} \int_{\mathcal{E}_1(K,j,n)} |g(x)|^K dx &\le \sum_{v\ge1}\int_{\mathcal{T}(K,j,v)} |g(x)|^K dx \le \sum_{v\ge1} m(\mathcal{T}(K,j,v))a(K,v)^K \\ &\le |I(K,j)|\exp(-K) \end{aligned}$$

by Lemma 3.4. $\square$

Lemma 3.10.

$$m(\mathcal{E}_2(K,j,n)) \le \exp(-\exp(3K))\,.$$

Proof. This follows from Definition 3.5 and Lemma 3.8. $\square$

Lemma 3.11. *We have for $n \ge n_0(K)$:*

$$\int_{\mathcal{E}_2(K,j,n)} |g(x)|^K dx \le (|j| + 1)^{-2}\exp(-K)$$

Proof. For $x \in \mathcal{Z}(K,j,v,w,n)$ we have:

$$|g(x)| \leq b(x,K,j,n) + w + 1 + |D(x,n)|,$$

where $b(x,K,j,n) := l(x) + L_0 - j + w + 1$. Thus

$$\int_{\mathcal{Z}(K,j,v,w,n)} |g(x)|^K dx \leq 2^K \left(\sup_{x \in I(K,j)} |b(x,K,j,n)|^K + |I(L_0,j)| l(x^{(j-1)})^K \exp\left(-\frac{C_1}{2}K^2 + (w+1)K\right)\right)$$
$$\times\, (m(\mathcal{T}(K,j,v) + m(\mathcal{V}(K,j,w,v,n)).$$

The result follows by summation over v and w. □

Definition 3.12. *We set*

$$x_0 := \exp\left(-\left\lfloor \frac{L_0}{100} \right\rfloor\right).$$

Lemma 3.13. *There is a contant $C_4 > 0$, such that*

$$\int_{x_0}^{1/2} |g(x)|^K dx \leq \Gamma(K+1)\exp(-C_4 K)\,.$$

Proof. We apply Lemma 2.22 of [9] with $L = L_0 := \lfloor K \rfloor + 1$. There is a constant $C_5^* > 0$, such that

$$\int_{x_0}^{1/2} |\mathcal{L}(x,n)|^{L_0} dx \leq \Gamma(L_0+1)\exp(-C_5^* L_0).$$

By Definitions 2.7, 2.8 and Lemma 2.10 we get

$$g(x) = \mathcal{L}(x,n) + H(x) + (-1)^{n+1} T^{n+1} \mathcal{W}(x)\,. \tag{3.2}$$

Thus by (3.2) using the notation

$$\|h\|_{L_0} := \left(\int_{x_0}^{1/2} |h(x)|^{L_0} dx \right)^{1/L_0}$$

we obtain

$$\left(\int_{x_0}^{1/2} |g(x)|^{L_0} dx \right)^{1/L_0} \leq (\Gamma(L_0+1)\exp(-C_5^* L_0))^{1/L_0} + \|H(x)\|_{L_0} + \|T^{n+1}\mathcal{W}(x)\|_{L_0}$$
$$= (\Gamma(L_0+1)\exp(-C_5^* L_0))^{1/L_0} \left(1 + O\left(\frac{1}{\Gamma(L_0+1)^{1/L_0}}\right)\right).$$

We obtain

$$\int_{x_0}^{1/2} |g(x,n)|^{L_0} dx = O(\Gamma(L_0+1)\exp(-C_5^* L_0)).$$

The result of Lemma 3.13 follows, since

$$\Gamma(L_0+1) = O(K\Gamma(K+1)).$$

□

Definition 3.14. *We set*

$$\mathcal{A} := ((0,x_0) \cap X) - \bigcup_{j \leq j_0} \mathcal{E}_1(K,j,n) - \bigcup_{j \leq j_0} \mathcal{E}_2(K,j,n).$$

Lemma 3.15. *There is a constant $C_5 > 0$, such that*

$$\int_0^{1/2} |g(x)|^K dx = \int_{\mathcal{A}} g(x)^K dx + O(\Gamma(K+1)\exp(-C_5 K)).$$

Proof. By Definition 3.14 we have

$$\int_0^{1/2} |g(x)|^K dx = \int_{\mathcal{A}} |g(x)|^K dx + \sum_{j \le j_0} \int_{\mathcal{E}_1(K,j,n)} |g(x)|^K dx$$

$$+ \sum_{j \le j_0} \int_{\mathcal{E}_2(K,j,n)} |g(x)|^K dx + \int_{x_0}^{1/2} |g(x)|^K dx. \tag{3.3}$$

From Lemmas 3.9, 3.11 and 3.13 we obtain

$$\int_{(0,1/2)\setminus\mathcal{A}} |g(x)|^K dx = O(\Gamma(K+1)\exp(-C_4 K)). \tag{3.4}$$

Therefore from (3.4) we get

$$\int_0^{1/2} |g(x)|^K dx = \int_{\mathcal{A}} |g(x)|^K dx + O(\Gamma(K+1)\exp(-C_4 K)). \tag{3.5}$$

By Definition 3.5 for $\mathcal{E}_1(K,j,n)$, $\mathcal{E}_2(K,j,n)$ and Lemma 2.10 we have $|g(x)| = g(x)$ for $x \in \mathcal{A}$. Lemma 3.15 follows from (3.5). □

Definition 3.16. *For $x \in X$ let*

$$R(x,n) := (D(x,n) + H(x) + (-1)^{n+1} T^{n+1} \mathcal{W}(x)) l(x)^{-1}.$$

Lemma 3.17. *There is a constant $C_6 > 0$, such that for $x \in \mathcal{A}$ we have*

$$R(x,n) = (\gamma - 2\pi) l(x)^{-1} + O(\exp(-C_6 K)).$$

Proof. By the Definition 3.5 for $\mathcal{E}_1(K,j,n)$, $\mathcal{E}_2(K,j,n)$ and Definition 3.14 for $\mathcal{A}$, we have for $x \in \mathcal{A}$

$$|D(x,n)| < \exp\left(-\frac{C_1}{2} K\right). \tag{3.6}$$

$$|T^{n+1}\mathcal{W}(x)| < \exp\left(-\frac{C_1}{2} K\right). \tag{3.7}$$

By Definition 2.8 we have

$$H(x) = 2\sum_{j \ge 0} (-1)^j \beta_{j-1}(x) F(\alpha_j(x)),$$

where

$$F(x) := \frac{x+1}{2} A(1) - A(x) - \frac{x}{2}\log x.$$

By Lemma 2.9 we have

$$A(x) = \frac{x}{2}\log\frac{1}{x} + \frac{1 + A(1)}{2} x + O(x^2).$$

From $\beta_{j-1} = \alpha_1(x) \cdots \alpha_{j-1}(x)$ with $\alpha_0(x) = x$, $|\alpha_l(x)| \le 1$ and $x \le x_0$ it follows that

$$H(x) = -A(1) + O\left(\exp\left(-\frac{K}{200}\right)\right). \tag{3.8}$$

□

In [1] it is proved at page 225 that

$$A(1) = \log 2\pi - \gamma. \tag{3.9}$$

Lemma 3.17 now follows from (3.6), (3.7), (3.8) and (3.9).

Lemma 3.18. *For $x \in \mathcal{A}$ we have*

$$|g(x)|^K = g(x)^K = l(x)^K \sum_{j=0}^{\infty} \binom{K}{j} R(x,n)^j \,.$$

Proof. By Lemma 2.10 we have for $x \in \mathcal{A}$:

$$g(x) = l(x)(1 + R(x,n)), \text{ where } |R(x,n)| < 1,$$

by Lemma 3.17. The result of Lemma 3.18 thus follows from the Binomial Theorem for real exponents. □

Lemma 3.19. *For $x \in \mathcal{A}$ we have*

$$g(x)^K = l(x)^K \left(\sum_{j=0}^{\lfloor K \rfloor} \binom{K}{j} R(x,n)^j + O(\exp(-K)) \right) \,.$$

Proof. From Definition 3.12 we have for $x \in (0, x_0)$:

$$|l(x)| \geq cK \text{ with an absolute constant c>0.}$$

By Definitions 3.5 and 3.14 we have for $x \in \mathcal{A}$:

$$|R(x,n)| \leq BK^{-1},$$

where $B > 0$ is an absolute constant. For $0 \leq j \leq \lfloor K \rfloor$ we have

$$\binom{K}{j} < \binom{\lfloor K \rfloor + 1}{j} \leq 2^{K+1}.$$

For $j = \lfloor K \rfloor + h$, $h \in \mathbb{N}$ we have

$$\binom{K}{j} \leq \binom{K}{\lfloor K \rfloor} \frac{2}{\lfloor K \rfloor + 1} \cdots \frac{h+1}{\lfloor K \rfloor + h} \leq 2^{K+1} \tag{3.7}$$

From (3.6) and (3.7) we obtain

$$\sum_{j > \lfloor K \rfloor} \binom{K}{j} R(x,n)^j \leq 2^{K+1} \frac{(BK^{-1})^{K-1}}{1 - (BK^{-1})} \,. \tag{3.8}$$

Lemma 3.19 follows from Lemma 3.18 and (3.8). □

Lemma 3.20. *There is an absolute constant $C_7 > 0$, such that for $K/2 < L \leq K$, $L \in \mathbb{R}$, we have:*

$$\int_{\mathcal{A}} l(x)^L dx = \Gamma(L+1)(1 + O(\exp(-C_7 K))) \,. \tag{3.9}$$

Proof. By Definition 3.14 we have:

$$\int_{(0,1/2)\setminus\mathcal{A}} l(x)^L dx = \sum_{j \leq j_0} \left(\int_{\mathcal{E}_1(K,j,n)} l(x)^L dx + \int_{\mathcal{E}_2(K,j,n)} l(x)^L dx \right) + \int_{x_0}^{1/2} l(x)^L dx \,. \tag{3.10}$$

There are absolute constants $c_1 > 0$, $c_{2,i} > 0$ (i=1,2) such that

$$\min_{x\in I(K,j)} l(x)^L \le \max_{x\in I(K,j)} l(x)^L \le c_1 \min_{x\in I(K,j)} l(x)^L$$

$$\min_{x\in\mathcal{E}_i(K,j,n)} l(x)^L \le \max_{x\in\mathcal{E}_i(K,j,n)} l(x)^L \le c_2 \min_{x\in\mathcal{E}_i(K,j,n)} l(x)^L.$$

Therefore there is an absolute constant $c_3 > 0$, such that

$$\int_{\mathcal{E}_i(K,j,n)} l(x)^L dx \le c_3 \left(\int_{I(K,j)} l(x)^L dx\right) m(\mathcal{E}_i(K,j,n)). \tag{3.11}$$

By summation over j and Lemma 3.6.

$$\sum_{j\le j_0} \int_{\mathcal{E}_1(K,j,n)} l(x)^L dx \le \Gamma(L+1)\exp(-K). \tag{3.12}$$

We have

$$\sum_{j\le j_0} \int_{\mathcal{E}_2(K,j,n)} l(x)^L dx = \Sigma^{(1)} + \Sigma^{(2)} \quad ,$$

where

$$\Sigma^{(1)} := \sum_{-\exp(e^K)\le j\le j_0} \int_{\mathcal{E}_2(K,j,n)} l(x)^L dx,$$

$$\Sigma^{(2)} := \sum_{j<-\exp(\exp(K))} \int_{\mathcal{E}_2(K,j,n)} l(x)^L dx.$$

For

$$-\exp(e^K) \le j \le j_0$$

we have by Lemma 3.10:

$$m(\mathcal{E}_2(K,j,n)) \le |I(K,j)| \exp(-\exp(3K)).$$

Therefore
(3.13)

$$\Sigma^{(1)} \le \left(\int_{\exp(-L_0-\exp(e^K))}^{\infty} l(x)^L dx\right) \exp(-\exp(3K)) \le \exp(-\exp(3K))\Gamma(L+1).$$

We have

$$\Sigma^{(2)} \le \int_0^{\exp(L_0-\exp(e^K))} l(x)^K dx \le \Gamma(L+1)\exp\left(-\exp\left(\frac{K}{2}\right)\right) \tag{3.14}$$

We also have

$$\int_{x_0}^{1/2} l(x)^L dx = \Gamma(L+1)\exp(-C_7' K) \tag{3.15}$$

for an appropriate constant $C_7' > 0$, if $K/2 < L \le K$.
Lemma 3.20 now follows from (3.9)–(3.15). $\square$

Lemma 3.21. *There is an absolute constant $C_8 > 0$, such that for $0 \le j \le \lfloor K \rfloor$ we have:*

$$\int_{\mathcal{A}} l(x)^K R(x,n)^j dx = D^j \left(\int_{\mathcal{A}} l(x)^{K-j} dx\right)(1 + O(\exp(-C_8 K)),$$

where $D := \gamma - \log 2\pi$.

Proof. By Lemma 3.17 we have

$$R(x,n) = Dl(x)^{-1} + Q(x), \tag{3.16}$$

where

$$Q(x) := O(\exp(-C_6 K))\,.$$

By the Binomial Theorem we have

(3.17)

$$\int_{\mathcal{A}} l(x)^K R(x,n)^j dx = D^j \int_{\mathcal{A}} l(x)^{K-j} dx + \sum_{h=1}^{j} \binom{j}{h} D^{j-h} \int_{\mathcal{A}} l(x)^{K-j+h} Q(x)^h dx.$$

We have

$$\int_{\mathcal{A}} l(x)^{K-j+h} Q(x)^h dx \le \Gamma(K-j+h+1)\exp(-C_6 hK),$$

where

$$\Gamma(K-j+h) \le K^h \Gamma(K-j+1)\,.$$

Therefore

$$\int_{\mathcal{A}} l(x)^{K-j+h} Q(x)^h dx \le (K\exp(-C_6 K)^h)\Gamma(K-j+1)$$

and Lemma 3.21 follows from (3.17). □

4. Proof of Theorem 1.1

By Lemma 3.15 we have

$$\int_0^{1/2} |g(x)|^K dx = \int_{\mathcal{A}} g(x)^K dx + O(\Gamma(K+1)\exp(-C_5 K)). \tag{3.18}$$

From Lemma 3.19 we obtain

$$\int_{\mathcal{A}} g(x)^K dx = \sum_{j=0}^{\lfloor K \rfloor} \binom{K}{j} \int_{\mathcal{A}} l(x)^K R(x,n)^j dx + O(\Gamma(K+1)\exp(-K)). \tag{3.19}$$

From Lemmas 3.20 and 3.21, formulas (3.18) and (3.19), we obtain:

(3.20)

$$\begin{aligned}\int_0^{1/2} |g(x)|^K dx = &\sum_{j \le K/2} D^j \binom{K}{j} \Gamma(K-j+1)(1+O(\exp(-C_7 K)) \\ &+ \sum_{K/2 < j \le K} \binom{K}{j} \int_{\mathcal{A}} l(x)^K R(x,n)^j dx + O(\Gamma(K+1)\exp(-K))\,.\end{aligned}$$

We have

$$\binom{K}{j} \Gamma(K-j+1) = \frac{1}{j!}\Gamma(K+1) \tag{3.21}$$

Therefore

$$\begin{aligned}\sum_{j \le K/2} D^j \binom{K}{j} \Gamma(K-j+1) &= \Gamma(K+1) \sum_{j=0}^{\infty} \frac{D^j}{j!}\,(1+O(\exp(-CK))) \\ &= \Gamma(K+1)e^D(1+O(\exp(-CK)))\end{aligned}$$

and thus Theorem 1.1 is proved. □

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